CHAPTER 8

TRIGONOMETRY 2

Compound, Multiple and Half-angle Formulas

A **compound angle** is an angle which is written as the sum or difference of two or more angles.

For example, $(A + B)$, $(A - B)$ and $(A - B - C)$ are compound angles.

A **multiple angle** is an angle which is written as a multiple of a single angle.

For example, $2A$, $2B$, 3θ are multiple angles.

A sub-multiple angle is an angle which is written as a fraction of a single angle.

For example, $\frac{A}{2}, \frac{B}{2}, \frac{\theta}{3}$ are sub-multiple angles.

We shall now derive formulas for trigonometric functions of these angles.

In deriving these formulas we make use of the following identities for all $A \in \mathbf{R}$.

1. $\cos^2 A + \sin^2 A = 1$	**2.** $\cos(-A) = \cos A$	**3.** $\sin(-A) = -\sin A$
4. $\tan A = \dfrac{\sin A}{\cos A}$	**5.** $\cos(\frac{\pi}{2} - A) = \sin A$	**6.** $\sin(\frac{\pi}{2} - A) = \cos A$

Example

Given $\cos(A - B) = \cos A \cos B + \sin A \sin B$, prove each of the following:

1. $\cos(A + B) = \cos A \cos B - \sin A \sin B$

2. $\cos 2A = \cos^2 A - \sin^2 A$

3. $\sin(A + B) = \sin A \cos B + \cos A \sin B$

4. $\sin(A - B) = \sin A \cos B - \cos A \sin B$

5. $\sin 2A = 2 \sin A \cos A$

6. $\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}$

7. $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

8. $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

9. $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

10. $\tan 6\theta = \dfrac{2 \tan 3\theta}{1 - \tan^2 3\theta}$

11. $\cos^2 A = \frac{1}{2}(1 + \cos 2A)$

12. $\sin^2 A = \frac{1}{2}(1 - \cos 2A)$

13. $\cos 2A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$

14. $\sin 2A = \dfrac{2 \tan A}{1 + \tan^2 A}$

Solution:

We derive each of these formulas based on the formula:

$\cos(A - B) = \cos A \cos B + \sin A \sin B$.

We then use each new formula to derive another formula.

1. **Prove:** $\cos(A + B) = \cos A \cos B - \sin A \sin B$

   Given: $\cos(A - B) = \cos A \cos B + \sin A \sin B$

   Proof: Replace B with $(-B)$ on both sides.

$$\cos[A - (-B)] = \cos A \cos(-B) + \sin A \sin(-B)$$
$$\cos(A + B) = \cos A \cos B + \sin A(-\sin B)$$
$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

2. **Prove:** $\cos 2A = \cos^2 A - \sin^2 A$

   Given: $\cos(A + B) = \cos A \cos B - \sin A \sin B$

   Proof: Replace B with A on both sides.

$$\cos(A + A) = \cos A \cos A + \sin A \sin A$$
$$\cos 2A = \cos^2 A - \sin^2 A$$

3. **Prove:** $\sin(A + B) = \sin A \cos B + \cos A \sin B$

   Given: $\cos(A - B) = \cos A \cos B + \sin A \sin B$

   Proof: Replace A with $(\frac{\pi}{2} - A)$ on both sides.

$$\cos[(\tfrac{\pi}{2} - A) - B] = \cos(\tfrac{\pi}{2} - A)\cos B + \sin(\tfrac{\pi}{2} - A)\sin B$$
$$\cos[\tfrac{\pi}{2} - (A + B) = \sin A \cos B + \cos A \sin B$$
$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

4. **Prove:** $\sin(A - B) = \sin A \cos B - \cos A \sin B$

   Given: $\sin(A + B) = \sin A \cos B + \cos A \sin B$

   Proof: Replace B with $(-B)$ on both sides.

$$\sin[A + (-B)] = \sin A \cos(-B) + \cos A \sin(-B)$$
$$\sin(A - B) = \sin A \cos B + \cos A(-\sin B)$$
$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

5. **Prove:** $\sin 2A = 2 \sin A \cos A$

   Given: $\sin(A + B) = \sin A \cos B + \cos A \sin B$

   Proof: Replace B with A on both sides.

$$\sin(A + A) = \sin A \cos A + \cos A \sin A$$
$$\sin 2A = 2 \sin A \cos A$$

6. Prove: $$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}$$

Given: $$\sin 2A = 2 \sin A \cos A$$

Proof: Replace A with $\frac{A}{2}$ on both sides.

$$\sin 2\left(\frac{A}{2}\right) = 2 \sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)$$

$$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}$$

7. Prove: $$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Given: $$\tan A = \frac{\sin A}{\cos A}$$

Proof: Replace A with $(A + B)$ on both sides.

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

$$\tan(A+B) = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \cos A \sin B}$$

$$\tan(A+B) = \frac{\dfrac{\sin A \cos B}{\cos A \cos B} + \dfrac{\cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B}{\cos A \cos B} - \dfrac{\sin A \sin B}{\cos A \cos B}} \quad \left(\begin{array}{l}\text{Divide top and bottom} \\ \text{by } \cos A \cos B\end{array}\right)$$

$$\tan(A+B) = \frac{\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}}{1 - \dfrac{\sin A}{\cos A} \cdot \dfrac{\sin B}{\cos B}}$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

8. Prove: $$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Given: $$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Proof: Replace B with $(-B)$ on both sides.

$$\tan(A + (-B)) = \frac{\tan A + \tan(-B)}{1 - \tan A \tan(-B)}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

9. Prove: $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

Given: $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

Proof: Replace B with A on both sides

$$\tan(A+A) = \frac{\tan A + \tan A}{1 - \tan A \tan A}$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

10. Prove: $\tan 6\theta = \dfrac{2\tan 3\theta}{1-\tan^2 3\theta}$

Given: $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

Proof: Replace A with 3θ on both sides

$$\tan 2(3\theta) = \frac{2\tan 3\theta}{1-\tan^2 3\theta}$$

$$\tan 6\theta = \frac{2\tan 3\theta}{1-\tan^2 3\theta}$$

11. Prove: $\cos^2 A = \frac{1}{2}(1+\cos 2A)$

Given: $\cos 2A = \cos^2 A - \sin^2 A$

Proof:

$$\cos 2A = \cos^2 A - (1-\cos^2 A)$$

$$\cos 2A = \cos^2 A - 1 + \cos^2 A$$

$$\cos 2A = 2\cos^2 A - 1$$

$$2\cos^2 A = 1 + \cos 2A$$

$$\cos^2 A = \tfrac{1}{2}(1+\cos 2A)$$

12. Prove: $\sin^2 A = \frac{1}{2}(1-\cos 2A)$

Given: $\cos 2A = \cos^2 A - \sin^2 A$

Proof:

$$\cos 2A = (1-\sin^2 A) - \sin^2 A$$

$$\cos 2A = 1 - \sin^2 A - \sin^2 A$$

$$\cos 2A = 1 - 2\sin^2 A$$

$$2\sin^2 A = 1 - \cos 2A$$

$$\sin^2 A = \tfrac{1}{2}(1-\cos 2A)$$

13. Prove: $\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

Proof:

$$\frac{1-\tan^2 A}{1+\tan^2 A} = \frac{1-\dfrac{\sin^2 A}{\cos^2 A}}{1+\dfrac{\sin^2 A}{\cos^2 A}}$$

$$= \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A}$$

(multiplying top and bottom by $\cos^2 A$)

$$= \frac{\cos 2A}{1}$$

$$= \cos 2A$$

14. Prove: $\sin 2A = \dfrac{2\tan A}{1+\tan^2 A}$

Proof:

$$\frac{2\tan A}{1+\tan^2 A} = \frac{2\dfrac{\sin A}{\cos A}}{1+\dfrac{\sin^2 A}{\cos^2 A}}$$

$$= \frac{2\sin A\cos A}{\cos^2 A + \sin^2 A}$$

(multiplying top and bottom by $\cos^2 A$)

$$= \frac{\sin 2A}{1}$$

$$= \sin 2A$$

Example

Prove that $\sin 3A = 3\sin A - 4\sin^3 A$.

Solution:

Proof:

$\sin 3A$

$= \sin(2A + A)$

$= \sin 2A\cos A + \cos 2A\sin A$

$= (2\sin A\cos A)\cos A + (\cos^2 A - \sin^2 A)\sin A$

$= 2\sin A\cos^2 A + \sin A\cos^2 A - \sin^3 A$

$= 3\sin A\cos^2 A - \sin^3 A$

$= 3\sin A(1 - \sin^2 A) - \sin^3 A$

$= 3\sin A - 3\sin^3 A - \sin^3 A$

$= 3\sin A - 4\sin^3 A$

Applications

Example

Express **(i)** sin 15° and **(ii)** tan 105° in surd form.

Solution:

First express each angle as a combination of 30°, 45° or 60°.
Then use the compound angle formulas on page 9 of the tables.

(i) $\sin 15°$

$$= \sin(45° - 30°)$$
$$= \sin 45° \cos 30° - \cos 45° \sin 30°$$
$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$$
$$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

(ii) $\tan 105°$

$$= \tan(60° + 45°)$$
$$= \frac{\tan 60° + \tan 45°}{1 - \tan 60° \tan 45°}$$
$$= \frac{\sqrt{3}+1}{1-(\sqrt{3})(1)}$$
$$= \frac{\sqrt{3}+1}{1-\sqrt{3}}$$

Example

If $\tan(A + B) = 3$ and $\tan B = 2$, find the value of $\tan A$.

Solution:

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$
$$3 = \frac{\tan A + 2}{1 - (\tan A)(2)} \qquad \text{[given } \tan(A+B) = 3,\ \tan B = 2]$$
$$3 = \frac{\tan A + 2}{1 - 2\tan A} \qquad \text{[multiply both sides by } (1 - 2\tan A)]$$
$$3 - 6\tan A = \tan A + 2$$
$$-7\tan A = -1$$
$$7\tan A = 1$$
$$\tan A = \frac{1}{7}$$

Alternatively,

$$\tan A = \tan[(A+B) - B] = \frac{\tan(A+B) - \tan B}{1 + \tan(A+B)(\tan B)} = \frac{3-2}{1+(3)(2)} = \frac{1}{7}$$

Example

If $\sin 2A = \frac{7}{25}$, $0 \leqslant A \leqslant \frac{\pi}{2}$, find $\tan A$, $\sin A$, and $\cos A$.

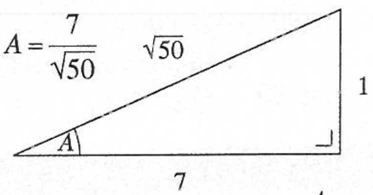
Solution:

$$\sin 2A = \frac{2\tan A}{1+\tan^2 A}$$
$$\frac{7}{25} = \frac{2\tan A}{1+\tan^2 A}$$
(let $t = \tan A$)
$$\frac{7}{25} = \frac{2t}{1+t^2}$$
$$7 + 7t^2 = 50t$$
$$7t^2 - 50t + 7 = 0$$
$$(7t-1)(t-7) = 0$$
$$7t - 1 = 0 \quad \text{or} \quad t - 7 = 0$$
$$t = \frac{1}{7} \quad \text{or} \quad t = 7$$
$$\tan A = \frac{1}{7} \quad \text{or} \quad \tan A = 7$$

1. $\tan A = \frac{1}{7}$

$\sin A = \frac{1}{\sqrt{50}}$

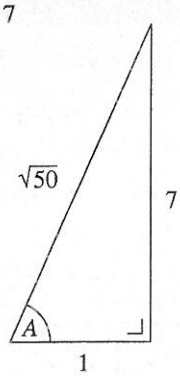
and $\cos A = \frac{7}{\sqrt{50}}$



2. $\tan A = \frac{7}{1}$

$\sin A = \frac{7}{\sqrt{50}}$

and $\cos A = \frac{1}{\sqrt{50}}$



Exercise 8.1

1. A and B are acute angles such that $\sin A = \frac{12}{13}$ and $\cos B = \frac{4}{5}$.

Without the use of tables or calculator, find the value of:

(i) $\cos A$ **(ii)** $\tan A$ **(iii)** $\sin B$ **(iv)** $\tan B$
(v) $\sin(A+B)$ **(vi)** $\cos(A+B)$ **(vii)** $\tan(A+B)$ **(viii)** $\sin 2A$

2. (i) If $\tan\theta = \frac{20}{21}$, $0 < \theta < \frac{\pi}{2}$, without using tables or calculator find the value of:

(a) $\sin 2\theta$ **(b)** $\cos 2\theta$ **(c)** $\tan 2\theta$.

(ii) If $\cos x = \frac{1}{\sqrt{5}}$, find the value of $\cos 2x$ without using tables or a calculator.

Express each of the following in surd form:

3. $\cos 75°$ **4.** $\sin 105°$ **5.** $\tan 75°$ **6.** $\cos 15°$

7. $\sin 165°$ **8.** $\cot 15°$ **9.** $\sec 15°$ **10.** $\operatorname{cosec} \frac{5\pi}{12}$

11. $\cos 25° \cos 20° - \sin 25° \sin 20°$ **12.** $\sin 70° \cos 10° - \cos 70° \sin 10°$

13. $\dfrac{\tan 80° - \tan 20°}{1 + \tan 80° \tan 20°}$

14. A and B are acute angles such that $\tan A = 4$ and $\tan(A + B) = 5$. Find $\tan B$.

15. A and B are acute angles such that $\tan B = \frac{1}{4}$ and $\tan(A - B) = 2$. Find **(i)** $\tan A$ **(ii)** $\sin 2A$.

16. A and B are acute angles such that $\tan A = \frac{3}{5}$ and $\tan B = \frac{1}{4}$. Find $(A + B)$.

17. $\tan A = \frac{1}{2}$, $180° < A < 360°$, $\tan B = -\frac{1}{3}$, $-90° < B < 90°$. Find $(A - B)$.

Verify each of the following:

18. $\sin(90° - A) = \cos A$

19. $\cos(90° + A) = -\sin A$

20. $\cos(180° - A) = -\cos A$

21. $\sin(180° - A) = \sin A$

22. $\tan(45° + A) = \dfrac{1 + \tan A}{1 - \tan A}$

23. $\sin\left(\dfrac{\pi}{2} + \alpha\right) - \sin\left(\dfrac{\pi}{2} - \alpha\right) = 0$

24. If $\cos 2A = \frac{1}{49}$, find the two values of $\cos A$ without using tables or calculator.

25. If $\cos 2A = \frac{12}{13}$, find the two possible values of $\tan A$.

26. A, B, C and D are acute angles such that $\tan A = \frac{1}{3}$, $\tan B = \frac{1}{5}$, $\tan C = \frac{1}{7}$ and $\tan D = \frac{1}{8}$.
Evaluate **(a)** $\tan(A + B)$ **(b)** $\tan(C + D)$.
Hence, or otherwise, find the value of $\tan(A + B + C + D)$.
Prove that $A + B + C + D = 45°$.

Prove each of the following:

27. $2\cos^2 A - \cos 2A - 1 = 0$

28. $\sin(A + B)\sin(A - B) = \cos^2 B + \sin^2 A - 1$

29. $\dfrac{\sin 2A}{1 + \cos 2A} = \tan A$

30. $\dfrac{1 - \cos 2A}{1 + \cos 2A} = \tan^2 A$

31. $\cos 3A = 4\cos^3 A - 3\cos A$

32. $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

33. Use the formula $\cos 2A = 2\cos^2 A - 1$, to write $\cos\dfrac{\pi}{8}$ in surd form.

Sum and Product Formulas

Changing products into sums and differences:

$$2\cos A\cos B = \cos(A + B) + \cos(A - B)$$
$$2\sin A\cos B = \sin(A + B) + \sin(A - B)$$
$$2\sin A\sin B = \cos(A - B) - \cos(A + B)$$
$$2\cos A\sin B = \sin(A + B) - \sin(A - B)$$

Changing sums and differences into products:

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\cos A - \cos B = -2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

These are given on page 9 of the tables. The proof of these 8 formulas is **not** required.

Example

(i) Express $\cos 3\theta \sin 5\theta$ as a sum or difference of two trigonometric functions.
(ii) Express $\sin 6\theta + \sin 4\theta$ as the product of two trigonometric functions.

Solution:

(i) $\sin 5\theta \cos 3\theta$ (larger angle first)

$A = 5\theta \qquad B = 3\theta$

$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

$\therefore\ \sin 5\theta \cos 3\theta$

$= \frac{1}{2}[\sin(5\theta + 3\theta) + \sin(5\theta - 3\theta)]$

$= \frac{1}{2}[\sin 8\theta + \sin 2\theta]$

(ii) $\sin 6\theta + \sin 4\theta$

$A = 6\theta \qquad B = 4\theta$

$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$\therefore\ \sin 6\theta + \sin 4\theta$

$= 2\sin\left(\frac{6\theta + 4\theta}{2}\right)\cos\left(\frac{6\theta - 4\theta}{2}\right)$

$= 2\sin 5\theta \cos\theta$

Example

(i) Find the exact value of $\sin 105° - \sin 15°$.

(ii) Prove that $\dfrac{\sin\theta - \sin 2\theta + \sin 3\theta}{\cos\theta - \cos 2\theta + \cos 3\theta} = \tan 2\theta$.

Solution:

(i)

$$\sin 105° - \sin 15° \qquad [A = 105°,\quad B = 15°]$$

$$= 2\cos\left(\frac{105° + 15°}{2}\right)\sin\left(\frac{105° - 15°}{2}\right) \qquad \left[\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)\right]$$

$$= 2\cos 60° \sin 45°$$

$$= 2 \times \frac{1}{2} \times \frac{1}{\sqrt{2}} \qquad \left[\cos 60° = \frac{1}{2},\quad \sin 45° = \frac{1}{\sqrt{2}}\right]$$

$$= \frac{1}{\sqrt{2}}$$

(ii)

$$\frac{\sin\theta - \sin 2\theta + \sin 3\theta}{\cos\theta - \cos 2\theta + \cos 3\theta}$$

$$= \frac{\sin 3\theta + \sin\theta - \sin 2\theta}{\cos 3\theta + \cos\theta - \cos 2\theta}$$

[linking the odd angles on top and bottom (or even angles if given) and putting the larger angle first in both cases]

$$= \frac{2\sin\left(\frac{3\theta+\theta}{2}\right)\cos\left(\frac{3\theta-\theta}{2}\right) - \sin 2\theta}{2\cos\left(\frac{3\theta+\theta}{2}\right)\cos\left(\frac{3\theta-\theta}{2}\right) - \cos 2\theta}$$

[using $\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$ on the top and $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$ on the bottom]

$$= \frac{2\sin 2\theta\cos\theta - \sin 2\theta}{2\cos 2\theta\cos\theta - \cos 2\theta}$$

$$= \frac{\sin 2\theta(2\cos\theta - 1)}{\cos 2\theta(2\cos\theta - 1)} \qquad \text{[factorising top and bottom]}$$

$$= \frac{\sin 2\theta}{\cos 2\theta} \qquad \text{[dividing top and bottom by } (2\cos\theta - 1)\text{]}$$

$$= \tan 2\theta$$

Exercise 8.2

Express each of the following as the product of two trigonometric functions:

1. $\sin 4\theta + \sin 2\theta$
2. $\cos 7\theta + \cos 5\theta$
3. $\cos 8\theta - \cos 2\theta$
4. $\sin 5\theta - \sin 3\theta$
5. $\cos 6\theta - \cos 2\theta$
6. $\cos \theta + \cos 7\theta$
7. $\sin \theta + \sin 3\theta$
8. $\cos \theta + \cos 5\theta$
9. $\sin 2\theta - \sin 8\theta$

Express each of the following as the sum or difference of two trigonometric functions:

10. $2 \sin 6\theta \cos 2\theta$
11. $2 \cos 3\theta \cos \theta$
12. $2 \cos 4\theta \cos \theta$
13. $2 \cos 6\theta \sin 3\theta$
14. $-2 \sin 4\theta \sin \theta$
15. $2 \cos 7\theta \sin 6\theta$
16. $\cos x \sin 5x$
17. $\sin 2A \sin A$
18. $\cos 3A \sin A$

Find the exact value of each of the following:

19. $\sin 75° - \sin 15°$
20. $\cos 105° - \cos 15°$
21. $\sin 255° - \sin 15°$
22. $2 \sin 75° \sin 105°$
23. $2 \cos 75° \cos 15°$
24. $\cos 37\tfrac{1}{2}° \sin 7\tfrac{1}{2}°$

Verify:

25. $\dfrac{\cos 80° - \cos 40°}{\sin 80° - \sin 40°} = -\sqrt{3}$
26. $\dfrac{\sin(120° + A) + \sin A}{\cos(60° - A) + \cos A} = \dfrac{1}{\sqrt{3}}$

Prove each of the following identities:

27. $\sin 3\theta + \sin \theta = 4 \sin \theta \cos^2 \theta$
28. $\cos 3\theta + \cos \theta = 4 \cos^3 \theta - 2 \cos \theta$
29. $\dfrac{\cos 5A + \cos 3A}{\sin 5A - \sin 3A} = \cot A$
30. $\dfrac{\cos 5A - \cos 3A}{\sin 3A - \sin A} = -2 \sin 2A$
31. $2 \cos\left(\dfrac{\pi}{4} + \theta\right)\cos\left(\dfrac{\pi}{4} - \theta\right) = \cos 2\theta$
32. $\dfrac{\cos 2\theta \cos \theta - \sin 4\theta \sin 3\theta}{\sin 8\theta \cos \theta - \sin 6\theta \cos 3\theta} = \cot 2\theta$

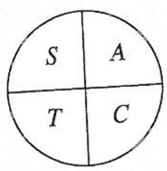
Trigonometric Equations I

Between 0° and 360° there may be two angles with the same trigonometric ratio.

For example, $\cos 120° = -\tfrac{1}{2}$ and $\cos 240° = -\tfrac{1}{2}$.

To solve a trigonometric equation do the following:

1. **Ignore the sign and calculate the related angle.**
2. **From the sign of the given ratio, decide in which quadrants the angles lie.**
3. **Using a rough diagram, state the angles between 0° and 360°.**



Maximum and minimum values for simple trigonometric equations

If $\sin\theta = k$, then,	$-1 \leqslant k \leqslant 1$	min. value = −1, max. value = 1
If $\cos\theta = k$, then,	$-1 \leqslant k \leqslant 1$	min. value = −1, max. value = 1
If $\tan\theta = k$, then,	$k \in \mathbf{R}$.	any value, $-\infty$ to ∞

Example

(i) Solve $\cos\theta = -\dfrac{1}{\sqrt{2}}$, $\quad 0 < \theta < 2\pi$.

(ii) Solve $\sin\theta = \dfrac{\sqrt{3}}{2}$, $\quad 0 < \theta < 2\pi$.

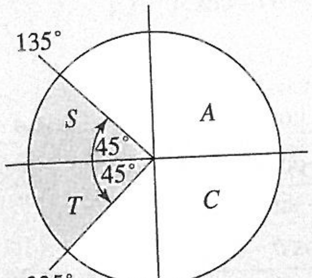
(iii) Solve $\sin^2\theta = \frac{1}{2}$, $\quad 0 \leqslant \theta \leqslant 360°$.

Solution:

(i) $\cos\theta = -\dfrac{1}{\sqrt{2}}$

related angle (ignore sign) = 45° or $\dfrac{\pi}{4}$

cos is negative in the 2nd and 3rd quadrants.



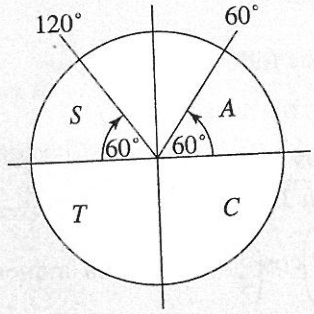
Thus, if $\cos\theta = -\dfrac{1}{\sqrt{2}}$, $\quad 0 < \theta < 2\pi$,

$\theta = 135°, 225°$ or $\theta = \dfrac{3\pi}{4}, \dfrac{5\pi}{4}$

(ii) $\sin\theta = \dfrac{\sqrt{3}}{2}$

related angle = 60° or $\dfrac{\pi}{3}$

sin is positive in the 1st and 2nd quadrants.



Thus, if $\sin\theta = \dfrac{\sqrt{3}}{2}$, $\quad 0 < \theta < 2\pi$,

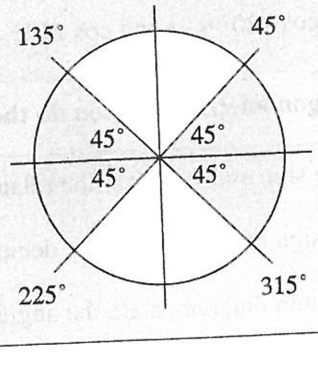
$\theta = 60°, 120°$ or $\theta = \dfrac{\pi}{3}, \dfrac{2\pi}{3}$

(iii) $\sin^2\theta = \frac{1}{2}$, $\quad 0 \leqslant \theta \leqslant 360°$

$\sin\theta = \pm\dfrac{1}{\sqrt{2}}$, related angle, $\theta = 45°$

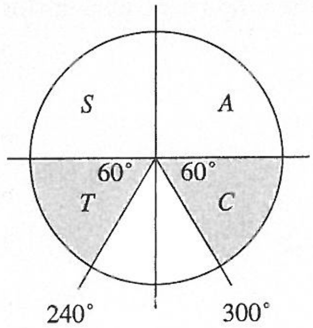
$\theta = 45°, 135°, 225°, 315°$

or $\quad \theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$



Example

Solve $\sin\left(x + \dfrac{\pi}{6}\right) = -\dfrac{\sqrt{3}}{2}$, $\quad 0 \leqslant x \leqslant 2\pi$.



Solution:

$$\sin\left(x + \frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}, \qquad 0 \leqslant x \leqslant 2\pi$$

let $\left(x + \dfrac{\pi}{6}\right) = \theta$

$\sin\theta = -\dfrac{\sqrt{3}}{2}$

related angle, $\theta = 60°$

sin is negative in the 3rd and 4th quadrants, thus $\theta = 240°$ or $300°$.

Thus, $x + \dfrac{\pi}{6} = 240°$ or $x + \dfrac{\pi}{6} = 300°$

$x + 30° = 240°$ $\qquad$ $x + 30° = 300°$

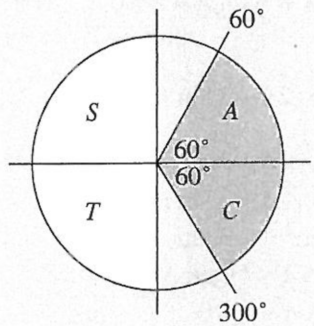
$x = 210°$ $\qquad$ $x = 270°$

or $x = \dfrac{7\pi}{6}$ or $x = \dfrac{3\pi}{2}$

We may have to solve equations involving multiple and sub-multiple angles.

Example

Solve $\cos 3A = \frac{1}{2}$, $\quad 0 \leqslant A \leqslant 360°$



Solution:

$\cos 3A = \frac{1}{2}$

related angle $= 60°$

cos is positive in the 1st and 4th quadrants.

Given: $0 \leqslant A \leqslant 360°$

$\therefore \quad 0 \leqslant 3A \leqslant 1080°$ (multiply each part by 3)

Thus, we need to go as far as $A + 1080°$

$\therefore \quad 3A = 60°, 60° + 360°, 60° + 720°, 60° + 1080°$ or $3A = 300°, 300° + 360°, 300° + 720°, 300° + 1080°$

$3A = 60°, 420°, 780°, 1140°$ or $3A = 300°, 660°, 1020°, 1380°$

$A = 20°, 140°, 260°, 380°$ or $A = 100°, 220°, 340°, 460°$

But we are given $0 \leqslant A \leqslant 360°$

Thus, $A = 20°, 100°, 140°, 220°, 260°, 340°$.

Exercise 8.3

Solve each of the following equations for $0 \leqslant \theta \leqslant 360°$.

1. $\tan \theta = 1$
2. $\cos \theta = \frac{1}{2}$
3. $\sin \theta = \dfrac{1}{\sqrt{2}}$
4. $\sin \theta = \dfrac{\sqrt{3}}{2}$
5. $\cos \theta = -\dfrac{1}{\sqrt{2}}$
6. $\sin \theta = -\dfrac{\sqrt{3}}{2}$
7. $\tan \theta = \sqrt{3}$
8. $\sin \theta = 0$
9. $\cos \theta = -1$
10. $\operatorname{cosec} \theta = -2$
11. $\cot \theta = -1$
12. $\sec \theta = -\dfrac{2}{\sqrt{3}}$
13. $\tan^2 \theta = \frac{1}{3}$
14. $4 \cos^2 \theta = 1$
15. $2 \sin^2 \theta - 1 = 0$
16. $\tan 3\theta = 1$
17. $\sin 2\theta = \frac{1}{2}$
18. $\cos \dfrac{\theta}{2} = -\dfrac{\sqrt{3}}{2}$
19. $\sin(\theta + 60°) = \frac{1}{2}$
20. $\cos(\theta - 45°) = \dfrac{\sqrt{3}}{2}$
21. $\tan(\theta + 30°) = -\sqrt{3}$
22. $\sin(2\theta + 30°) = \frac{1}{2}$
23. $\cos(3\theta + 15°) = \frac{1}{2}$
24. $\tan\left(\dfrac{\theta}{2} + 60°\right) = -\dfrac{1}{\sqrt{3}}$

In questions 25–27, give your answers correct to one place of decimals:

25. $100 \sin \theta = 43$
26. $4 \tan \theta = 5$
27. $3 \cos \theta = -1$

Trigonometric Equations 2

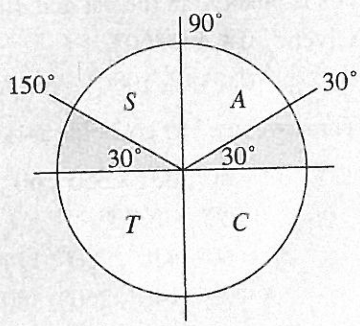
More complicated trigonometric equations can usually be reduced to one or more simple trigonometric equations by factorising or rearranging.

Example

Solve $\cos 2A + 3 \sin A - 2 = 0, \qquad 0 \leqslant A \leqslant 360°$.

Solution:

$$\cos 2A + 3 \sin A - 2 = 0$$
$$\downarrow$$
$$(1 - 2\sin^2 A) + 3 \sin A - 2 = 0 \qquad [\cos 2A = 1 - 2\sin^2 A]$$
$$-2\sin^2 A + 3 \sin A - 1 = 0$$
$$2\sin^2 A - 3 \sin A + 1 = 0$$
$$(2 \sin A - 1)(\sin A - 1) = 0 \qquad \text{[factorise]}$$
$$2 \sin A - 1 = 0 \quad \text{or} \quad \sin A - 1 = 0$$
$$\sin A = \tfrac{1}{2} \quad \text{or} \quad \sin A = 1$$



$\sin A = \frac{1}{2}$, related angle $= 30°$

sin is positive in the 1st and 2nd quadrants.

Thus $A = 30°$ or $150°$.

$\sin A = 1$

$A = 90°$

Thus, $A = 30°, 90°, 150°$

Example

Solve $\sqrt{2}\sin\theta\cos\theta + \cos\theta = 0, \qquad 0 \leqslant \theta \leqslant 360°.$

Solution:

$$\sqrt{2}\sin\theta\cos\theta + \cos\theta = 0$$

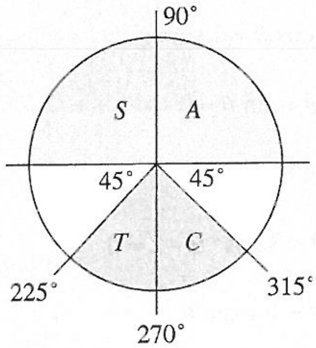
$$\cos\theta(\sqrt{2}\sin\theta + 1) = 0 \qquad \text{[take out common factor } \cos\theta]$$

$$\cos\theta = 0 \quad \text{or} \quad \sqrt{2}\sin A + 1 = 0$$

$$\cos\theta = 0 \quad \text{or} \quad \sin A = -\frac{1}{\sqrt{2}}$$

$$\cos\theta = 0$$

$$\theta = 90°, 270°$$



$\sin A = -\frac{1}{\sqrt{2}}$, related angle = 45°.

sin is negative in the 3rd and 4th quadrants.

$A = 225°$ or $315°$

Thus, $A = 90°, 225°, 270°, 315°$.

Example

Solve $\sin 5\theta + \sin 3\theta = 0, \qquad 0 \leqslant \theta \leqslant 180°.$

Solution:

$$\sin 5\theta + \sin 3\theta = 0$$

$$2\sin\left(\frac{5\theta + 3\theta}{2}\right)\cos\left(\frac{5\theta - 3\theta}{2}\right) = 0 \qquad \left[\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\right]$$

$$2\sin 4\theta\cos\theta = 0$$

$$\sin 4\theta\cos\theta = 0$$

$$\sin 4\theta = 0 \quad \text{or} \quad \cos\theta = 0$$

$$4\theta = 0°, 180°, 360°, 540°, 720° \quad \text{or} \quad \theta = 90°$$

$$\theta = 0°, 45°, 90°, 135°, 180° \quad \text{or} \quad \theta = 90°$$

Thus, $\theta = 0°, 45°, 90°, 135°, 180°$.

Note: As we are given $0 \leqslant \theta \leqslant 180°$, then $0 \leqslant 4\theta \leqslant 720°$.
Thus, we need to go as far as 720° for $\sin 4\theta = 0$.

Exercise 8.4

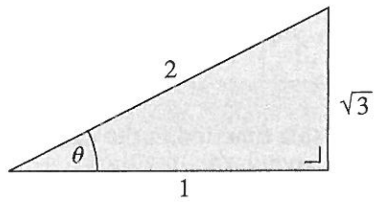
Solve each of the following equations for $0 \leqslant \theta \leqslant 360°$:

1. $(2\cos\theta - 1)(\cos\theta + 1) = 0$
2. $\sin\theta(2\sin\theta - 1) = 0$
3. $\tan^2\theta + \tan\theta = 0$
4. $2\cos^2\theta - \cos\theta = 0$
5. $2\sin^2\theta - \sin\theta - 1 = 0$
6. $\sqrt{3}\sin\theta = 2\cos^2\theta - 2$
7. $3 - 3\cos\theta = 2\sin^2\theta$
8. $\sqrt{2}\sin A\cos A - \cos A = 0$
9. $\cos 2\theta + \sin\theta = 0$
10. $\sin 2\theta + \sin\theta = 0$
11. $\cos 4\theta = \cos 2\theta$
12. $\cos 2\theta = 1 - 2\sin\theta$
13. $\sin^2\theta + 3\cos^2\theta - 2 = 0$
14. $4\cos^3\theta - \cos\theta = 0$
15. $\tan^2\theta = 1 + \sec\theta$
16. $2\sin\theta + 1 = \operatorname{cosec}\theta$
17. Show that $\sin(\theta + 30°) = \frac{1}{2}(\sqrt{3}\sin\theta + \cos\theta)$.
    Hence, solve $\sin(\theta + 30°) = 2\cos\theta, \quad 0 \leqslant \theta \leqslant 2\pi$.
18. Show that $\sin 3\theta + \sin\theta = 2\sin 2\theta\cos\theta$.
    Hence, show that $\sin 3\theta + \sin\theta + \sin 2\theta = \sin 2\theta(2\cos\theta + 1)$
    Hence, solve $\sin 3\theta + \sin\theta + \sin 2\theta = 0, \quad 0 < \theta < 180°$.
19. Prove that $\sin 2A = \dfrac{2\tan A}{1 + \tan^2 A}$ and hence solve $\sin 2A - \tan A = 0, \quad 0 \leqslant A \leqslant 180°$.
20. Show that $\sin 5x - \sin x = 2\cos 3x \sin 2x$.
    Hence, solve $\sin 5x - \sin x + \sqrt{3}\cos 3x = 0, \quad \pi \leqslant x \leqslant 2\pi$.
21. Prove that $\cos 3A = 4\cos^3 A - 3\cos A$, and hence solve $\cos 3A + 2\cos A = 0, \quad 0 \leqslant A \leqslant 180°$.
22. $x = 0°$ and $x = 60°$ are two solutions of the equation $a\sin^2 x + b\cos x - 3 = 0$, where $a, b \in \mathbf{N}$.
    Find the value of a and the value of b.
    Using these values of a and b, find all the solutions of the equation where $0° \leqslant x \leqslant 360°$.
23. Solve $\dfrac{1}{\sqrt{3}}\sin x - \cos\dfrac{x}{2} = 0, \quad 0 \leqslant x \leqslant 2\pi$.

Inverse Trigonometric Functions

We now consider the reverse process of determining an angle given the value of one of the trigonometric ratios.
Consider the right-angled triangle on the right.

Ratio	Angle
$\sin\theta = \dfrac{\sqrt{3}}{2}$	$\theta = \sin^{-1}\dfrac{\sqrt{3}}{2}$
$\cos\theta = \dfrac{1}{2}$	$\theta = \cos^{-1}\dfrac{1}{2}$
$\tan\theta = \sqrt{3}$	$\theta = \tan^{-1}\sqrt{3}$



However, a problem arises with the $\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$ notation.
Consider the following equations:

If $\sin\theta = \frac{1}{2}$, then $\theta = \sin^{-1}\frac{1}{2} = \ldots, -330°, -210°, 30°, 150°, 390°, 510°, \ldots$

If $\tan\theta = 1$, then $\theta = \tan^{-1} 1 = \ldots, -315°, -135°, 45°, 315°, 405°, 675°, \ldots$

From this we can see that if there is no restriction on the value of θ, then the equations $\sin\theta = \frac{1}{2}$ and $\tan\theta = 1$ have an infinite number of solutions.

To overcome this problem, the value of θ is restricted to the range $-90° \leqslant \theta \leqslant 90°$ or $-\dfrac{\pi}{2} \leqslant \theta \leqslant \dfrac{\pi}{2}$.

The angles within this range are often called the '**principal values**'.
Using these restrictions for θ we always obtain a single value for our answer.

Thus, $\sin^{-1}\dfrac{\sqrt{3}}{2} = 60°$, not also 120°, as 120° in not in the range −90° to 90°.

Note: $\sin^{-1} x$ is pronounced '**inverse sine *x***' or '**sine inverse *x***' or '**arc sine *x***'.

Domain and range of the 'inverse trigonometric functions':

$\sin^{-1} x$:	Domain $= [-1, 1]$,	Range $= \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$
$\tan^{-1} x$:	Domain $= [-\infty, \infty]$,	Range $= \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$
$\cos^{-1} x$:	Domain $= [-1, 1]$,	Range $= [0, \pi]$

Example

If $f(x) = \sin^{-1} x$, copy and complete the table below giving $\sin^{-1} x$ in terms of π.

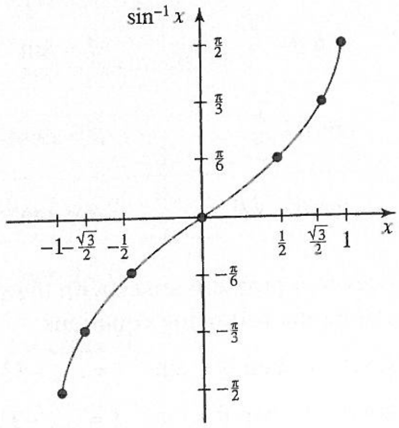
x	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1
$\sin^{-1} x$		$-\frac{\pi}{3}$		0			

Draw the graph of this function in the domain $[-1, 1]$.

Solution:

Completed table.

x	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1
$\sin^{-1} x$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$



Example

Evaluate: $\sin\left(\cos^{-1}\frac{8}{17} - \sin^{-1}\frac{12}{13}\right)$

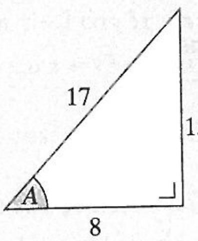
Solution:

$\cos^{-1}\frac{8}{17}$ and $\sin^{-1}\frac{12}{13}$ are angles.

let $\quad A = \cos^{-1}\frac{8}{17}$

$\cos A = \frac{8}{17}$

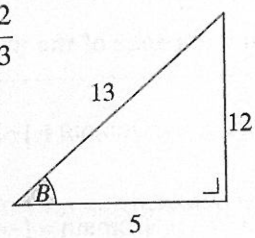
$\sin A = \frac{15}{17}$



let $\quad B = \sin^{-1}\frac{12}{13}$

$\sin B = \frac{12}{13}$

$\cos B = \frac{5}{13}$



$\sin\left(\cos^{-1}\frac{8}{17} - \sin^{-1}\frac{12}{13}\right)$

$= \sin(A - B)$ $\quad \left(A = \cos^{-1}\frac{8}{17},\ B = \sin^{-1}\frac{12}{13}\right)$

$= \sin A \cos B - \cos A \sin B$

$= \frac{15}{16}\cdot\frac{5}{13} - \frac{8}{17}\cdot\frac{12}{13}$

$= \frac{75}{221} - \frac{96}{221} = -\frac{21}{221}$

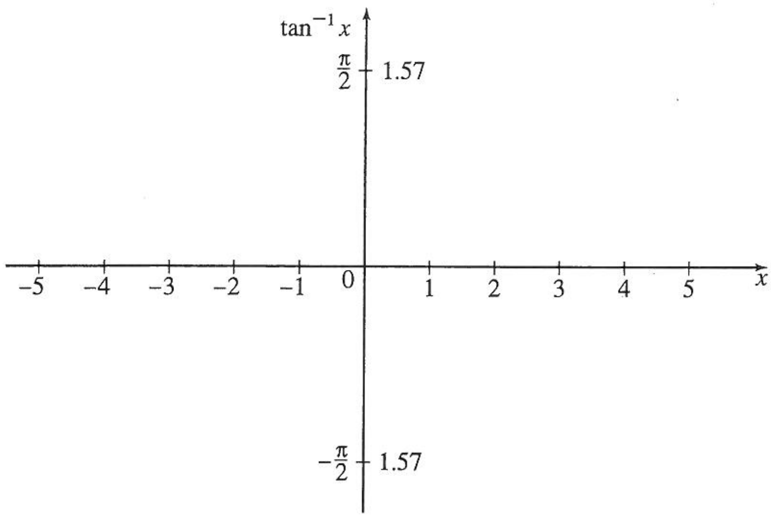
Exercise 8.5

Find the principal value of each of the following:

1. $\sin^{-1}\frac{1}{2}$
2. $\tan^{-1} 1$
3. $\cos^{-1}\frac{1}{\sqrt{2}}$
4. $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$
5. $\tan^{-1}(-\sqrt{3})$
6. $\sin^{-1} 0$
7. $\tan^{-1}(-1)$
8. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
9. $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$
10. $\sin^{-1}\left(-\frac{1}{2}\right)$

Drawing an appropriate right-angled triangle will help in the following problems.
Evaluate each of the following:

11. $\tan\left(\sin^{-1}\frac{3}{5}\right)$
12. $\cos\left(\sin^{-1}\frac{5}{13}\right)$
13. $\tan\left(\sin^{-1}\frac{\sqrt{3}}{2}\right)$
14. $\sin^2\left(\tan^{-1}\frac{3}{4}\right)$
15. $\tan^2\left(\sin^{-1}\frac{15}{17}\right)$
16. $\sin\left(2\cos^{-1}\frac{8}{17}\right)$
17. $\sin\left(\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{4}{5}\right)$
18. $\tan\left(\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{5}{13}\right)$
19. $\cos\left(\sin^{-1}\frac{3}{5}+\cos^{-1}\frac{12}{13}\right)$
20. $\sin\left(\sin^{-1}\frac{1}{\sqrt{5}}+\sin^{-1}\frac{1}{\sqrt{10}}\right)$
21. Prove that $\sin\left(\cos^{-1}\frac{7}{25}\right)=\sin\left(2\tan^{-1}\frac{3}{4}\right)$.
22. Prove that **(i)** $\sin(\tan^{-1} x)=\frac{x}{\sqrt{1+x^2}}$ **(ii)** $2\tan^{-1} x = \sin^{-1}\frac{2x}{1+x^2}$.
23. For $0 \leqslant x \leqslant 1$, prove that $\sin^{-1} x+\cos^{-1} x = \frac{\pi}{2}$.
24. Make sure your calculator is in radian mode.
    If $f(x)=\tan^{-1} x$, x in radians, draw the graph of f in the domain $-5 \leqslant x \leqslant 5$.



Limits of Trigonometry Functions

Let us consider the value of the expression $\frac{\sin\theta}{\theta}$ as θ approaches 0 (θ in radians).

θ (in radians)	**$\sin\theta$**	**$\frac{\sin\theta}{\theta}$**
1	0.8414709848	0.8414709848
0.5	0.4794255386	0.9588510772
0.1	0.0998334166	0.9983341665
0.01	0.0099998333	0.9999833334
0.001	0.0009999998	0.9999998333

As θ approaches 0, the expression $\frac{\sin\theta}{\theta}$ approaches 1.

This is written: $\boxed{\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1}$ (θ in radians)

The result can be extended to multiple and sub-multiple angles:

$$\boxed{\lim_{\theta\to 0}\frac{\sin k\theta}{k\theta} = 1 \quad \text{and} \quad \lim_{\theta\to 0}\frac{k\theta}{\sin k\theta} = 1, \quad k \in \mathbf{R}}$$

Similarly, where θ is in radians:

$$\boxed{\lim_{\theta\to 0}\frac{\tan k\theta}{k\theta} = 1 \quad \text{and} \quad \lim_{\theta\to 0}\frac{k\theta}{\tan k\theta} = 1, \quad k \in \mathbf{R}}$$

Note: $\lim_{\theta\to 0}\cos\theta = 1$.

Example

Evaluate: (i) $\lim_{\theta \to 0} \frac{\sin 4\theta + \sin 2\theta}{\theta}$ (ii) $\lim_{\theta \to 0} \frac{\tan 4\theta}{5\theta}$

Solution:

(i) $\frac{\sin 4\theta + \sin 2\theta}{\theta}$

$= \frac{\sin 4\theta}{\theta} + \frac{\sin 2\theta}{\theta}$

$= 4\left(\frac{\sin 4\theta}{4\theta}\right) + 2\left(\frac{\sin 2\theta}{2\theta}\right)$

$\therefore \lim_{\theta \to 0}\left(\frac{\sin 4\theta + \sin 2\theta}{\theta}\right)$

$= 4\left(\lim_{\theta \to \theta}\left(\frac{\sin 4\theta}{4\theta}\right)\right) + 2\left(\lim_{\theta \to \theta}\left(\frac{\sin 2\theta}{2\theta}\right)\right)$

$= 4(1) + 2(1)$

$= 4 + 2 = 6$

(ii) $\frac{\tan 4\theta}{5\theta}$

$= \frac{\tan 4\theta}{\theta} \cdot \frac{1}{5}$

$= \frac{\tan 4\theta}{4\theta} \cdot \frac{4}{5}$

$\therefore \lim_{\theta \to 0} \frac{\tan 4\theta}{5\theta}$

$= \frac{4}{5}\left(\lim_{\theta \to 0} \frac{\tan 4\theta}{4\theta}\right)$

$= \tfrac{4}{5}(1)$

$= \tfrac{4}{5}$

Sometimes we have to change sums and differences of trigonometric functions to products (using page 9 of the tables).

Example

Evaluate: $\lim_{x \to 0} \frac{\cos 5x - \cos 3x}{\cos 4x - \cos 2x}$

Solution:

$\frac{\cos 5x - \cos 3x}{\cos 4x - \cos 2x}$

$= \frac{-2 \sin\left(\frac{5x+3x}{2}\right)\sin\left(\frac{5x-3x}{2}\right)}{-2 \sin\left(\frac{4x+2x}{2}\right)\sin\left(\frac{4x-2x}{2}\right)}$

$= \frac{-2 \sin 4x \sin x}{-2 \sin 3x \sin x}$

$= \frac{\sin 4x}{\sin 3x}$

$= \frac{\sin 4x}{1} \cdot \frac{1}{\sin 3x}$

$= \frac{\sin 4x}{4x} \cdot \frac{3x}{\sin 3x} \cdot \frac{4}{3}$

From page 9 of the tables:
$\cos A - \cos B = -2 \sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

$\therefore \lim_{x \to 0} \frac{\cos 5x - \cos 3x}{\cos 4x - \cos 2x}$

$= \lim_{x \to 0} \frac{\sin 4x}{4x} \cdot \frac{3x}{\sin 4x} \cdot \frac{4}{3}$

$= (1)(1)(\tfrac{4}{3})$

$= \tfrac{4}{3}$

Exercise 8.6

Evaluate each of the following limits:

1. $\lim_{x\to 0} \frac{\sin 3x}{x}$

2. $\lim_{x\to 0} \frac{\sin 5x}{x}$

3. $\lim_{x\to 0} \frac{x}{\sin 2x}$

4. $\lim_{x\to 0} \frac{x}{\sin 4x}$

5. $\lim_{x\to 0} \frac{\sin 3x}{4x}$

6. $\lim_{x\to 0} \frac{\sin 3x}{\sin 2x}$

7. $\lim_{x\to 0} \frac{\tan 8x}{x}$

8. $\lim_{x\to 0} \frac{\tan 6x}{2x}$

9. $\lim_{x\to 0} \frac{\sin^2 x}{2x^2}$

10. $\lim_{x\to 0} \frac{\sin^2 x}{3x^2}$

11. $\lim_{x\to 0} \frac{\sin x \tan x}{x^2}$

12. $\lim_{x\to 0} \frac{\tan^2 2x}{3x^2}$

13. $\lim_{x\to 0} \frac{\sin 4x + \sin 2x}{x}$

14. $\lim_{x\to 0} \frac{\sin 8x - \sin 2x}{x}$

15. $\lim_{x\to 0} \frac{\cos 4x - \cos 2x}{\cos 5x - \cos 3x}$

16. $\lim_{x\to 0} \frac{x \sin x}{\sin 3x + \sin x}$

17. $\lim_{x\to 0} \frac{\sin 3x \tan 2x}{x^2}$

18. $\lim_{x\to 0} \frac{4x}{\sin 4x + \sin 2x}$

19. $\lim_{x\to 0} \frac{\sin x^2}{x \tan x}$

20. $\lim_{x\to 0} \frac{x \sin 2x}{2 - 2\cos^2 x}$

21. $\lim_{x\to 0} \frac{x \sin x}{1 - \cos x}$

22. Prove that $\cos 2x = 1 - 2\sin^2 x$.

    Hence, show that $\lim_{x\to 0} \frac{2x^2}{1 - \cos x} = 4$.