

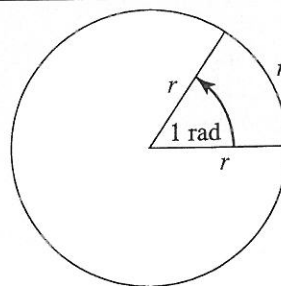
CHAPTER 7

TRIGONOMETRY I

Radian Measure

In more advanced trigonometry, and always in calculus, angles are measured in radians.

One radian is the measure of the angle at the centre of a circle subtended by an arc equal in length to the radius of the circle.



Note: Radians are often called '**circular measure**' and are denoted by rads.

The number of radians in one complete revolution is given by the ratio:

$$\frac{\text{circumference of a circle}}{\text{length of its radius}} = \frac{2\pi r}{r} = 2\pi \text{ radians}$$

Relationship between radians and degrees

One complete revolution = 2π radians = 360° , thus:

$$2\pi \text{ radians} = 360^\circ \quad \text{or} \quad \pi \text{ radians} = 180^\circ$$

Note: Very often the word radian is not used, thus we can write $\pi = 180^\circ$, where π means ' π radians' ($\pi \neq 180^\circ$).

The more common angles used are given in the following table:

Degrees	0°	30°	45°	60°	90°	120°	150°	180°	270°	360°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

These can be calculated from the equation π radians = 180° .

Note: $1 \text{ radian} = \frac{180^\circ}{\pi} \approx 57.3^\circ$ (correct to one decimal places).

Example ▼

Convert: (i) 225° to radians (ii) $\frac{5\pi}{3}$ radians to degrees.

Solution:

(i) $180^\circ = \pi$ radians

$$1^\circ = \frac{\pi}{180} \text{ radians}$$

$$225^\circ = 225 \times \frac{\pi}{180} \text{ radians}$$

$$225^\circ = \frac{5\pi}{4} \text{ radians}$$

(ii) $\frac{5\pi}{3}$ radians

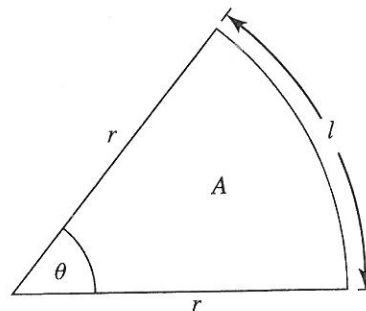
$$= \frac{5(180^\circ)}{3}$$

(put in $\pi = 180^\circ$)

$$= 300^\circ$$

The diagram shows a sector of a circle of radius r , angle θ , arc length l and area A .

The length of the arc l and the area of the sector A may be found by multiplying the length of the circumference and the area of the circle by $\frac{\theta}{2\pi}$.



Note: $\frac{\theta}{2\pi}$ is the fraction of the circle required.

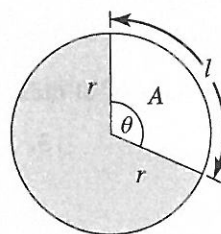
Length of arc, $l = \frac{\theta}{2\pi} \times 2\pi r = r\theta$

Area of sector, $A = \frac{\theta}{2\pi} \times \pi r^2 = \frac{1}{2}r^2\theta$

Arc length, $l = r\theta$

Area of sector, $A = \frac{1}{2}r^2\theta$

(θ in radians)



These formulas are on page 9 of the tables.

Each formula contains three variables and we are usually given two of these variables and asked to find the third one.

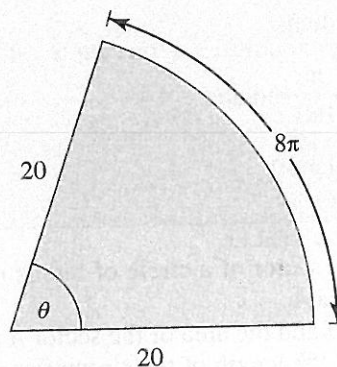
Example ▼

- (i) The radius of a circle is 20 cm. Find the angle subtended at the centre by an arc of length 8π cm.
- (ii) The area of a sector of a circle of radius r is 30 cm^2 . If the angle subtended at the centre of the circle by this sector is $\frac{5}{3}$ radians, calculate r .

Solution:

- (i) **Given:** $r = 20$, $l = 8\pi$, find θ .

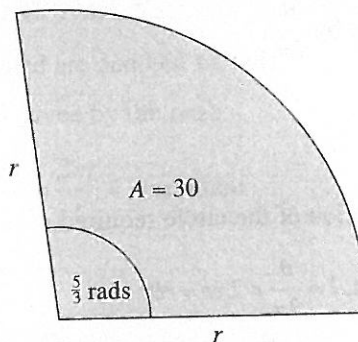
$$\begin{aligned} l &= r\theta \\ 8\pi &= 20\theta \\ 2\pi &= 5\theta \\ \frac{2\pi}{5} &= \theta \end{aligned}$$



- (ii) **Given:** $A = 30$, $\theta = \frac{5}{3}$ radians, find r .

$$\begin{aligned} A &= \frac{1}{2}r^2\theta \\ 30 &= \frac{1}{2}r^2 \times \frac{5}{3} \\ 30 &= \frac{5}{6}r^2 \\ 180 &= 5r^2 \\ 36 &= r^2 \\ 6 &= r \end{aligned}$$

$$\therefore r = 6 \text{ cm}$$



Exercise 7.1 ▼

Express each of the following number of radians in degrees:

- | | | | | |
|---------------------|---------------------|---------------------|----------------------|-----------------------|
| 1. π | 2. $\frac{\pi}{6}$ | 3. $\frac{\pi}{4}$ | 4. $\frac{2\pi}{3}$ | 5. $\frac{3\pi}{5}$ |
| 6. $\frac{4\pi}{3}$ | 7. $\frac{5\pi}{4}$ | 8. $\frac{4\pi}{9}$ | 9. $\frac{5\pi}{18}$ | 10. $\frac{11\pi}{6}$ |

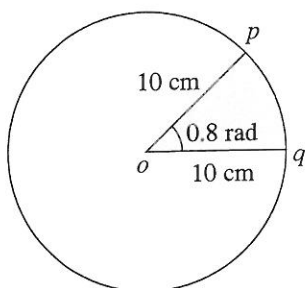
Express each of the following angles in radians, leaving π in your answers:

- | | | | | |
|-----------------|-----------------|-----------------|-----------------|---------------------------|
| 11. 30° | 12. 45° | 13. 60° | 14. 90° | 15. 120° |
| 16. 150° | 17. 210° | 18. 240° | 19. 135° | 20. 450° |
| 21. 390° | 22. 72° | 23. 288° | 24. 105° | 25. $22\frac{1}{2}^\circ$ |

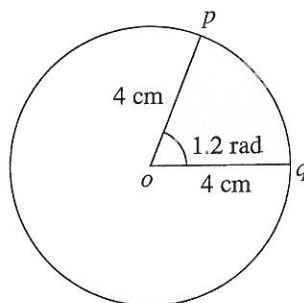
In each of the following find:

- (i) the length of the minor arc pq (ii) the area of the corresponding minor sector opq

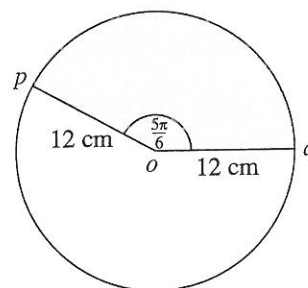
26.



27.



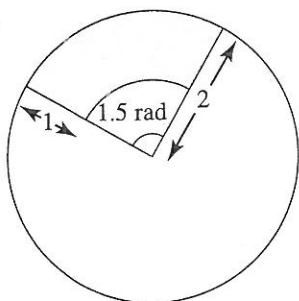
28.



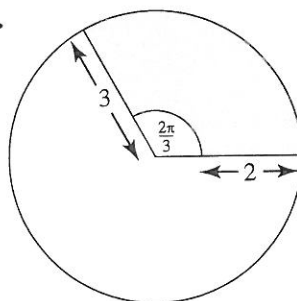
In each case below, find:

- (i) the area of the shaded region
(ii) the perimeter of the shaded region.

29.



30.



31. The radius of a circle is 12 cm. Find the angle subtended at the centre by an arc of length 16π cm.
32. Find the area of a sector of a circle of radius 20 cm if the arc of the sector subtends an angle of $\frac{2\pi}{5}$ at the centre.
33. The area of a sector of a circle, of radius r , is 24π cm². If the angle subtended at the centre of the circle by this sector is $\frac{3\pi}{4}$, calculate r , the radius of the circle.

Trigonometric Ratios

The six basic trigonometric ratios (or fractions) for a right-angled triangle are defined as follows for all angles $0^\circ < \theta < 90^\circ$, (or $0 < \theta < \frac{\pi}{2}$).

$$\sin \theta = \frac{O}{H}$$

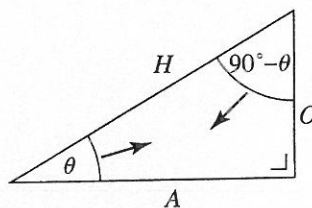
$$\cos \theta = \frac{A}{H}$$

$$\tan \theta = \frac{O}{A}$$

$$\operatorname{cosec} \theta = \frac{H}{O}$$

$$\sec \theta = \frac{H}{A}$$

$$\cot \theta = \frac{A}{O}$$



From this we can see that:

$$\operatorname{cosec} \theta = \frac{1}{\sin \theta}$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\sin(90^\circ - \theta) = \frac{A}{H} = \cos \theta$$

$$\sec \theta = \frac{1}{\cos \theta}$$

$$\cot \theta = \frac{1}{\tan \theta} = \frac{\cos \theta}{\sin \theta}$$

$$\cos(90^\circ - \theta) = \frac{O}{H} = \sin \theta$$

$$\tan(90^\circ - \theta) = \cot \theta$$

Note: These ratios hold for all values of $\theta \in \mathbf{R}$, not just for $0 < \theta < 90^\circ$.

sec, cosec, cot are short for secant, cosecant and cotangent, respectively.

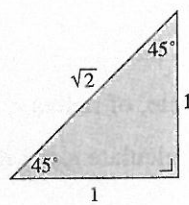
Special Angles: $45^\circ\left(\frac{\pi}{4}\right)$, $60^\circ\left(\frac{\pi}{3}\right)$, $30^\circ\left(\frac{\pi}{6}\right)$

There are three special angles whose sine, cosine and tangent ratios can be expressed as simple fractions or surds.

$$\sin 45^\circ = \frac{1}{\sqrt{2}}$$

$$\cos 45^\circ = \frac{1}{\sqrt{2}}$$

$$\tan 45^\circ = 1$$



$$\sin 60^\circ = \frac{\sqrt{3}}{2}$$

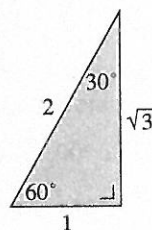
$$\cos 60^\circ = \frac{1}{2}$$

$$\tan 60^\circ = \sqrt{3}$$

$$\sin 30^\circ = \frac{1}{2}$$

$$\cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{1}{\sqrt{3}}$$



These ratios can be used instead of a calculator.

Trigonometric Ratios for Any Angle The Unit Circle

The unit circle has its centre at the origin (0, 0) and the length of the radius is 1.

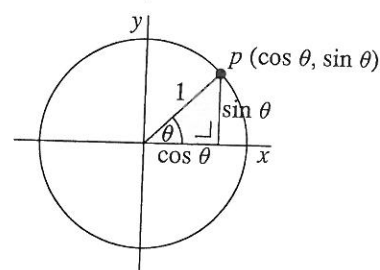
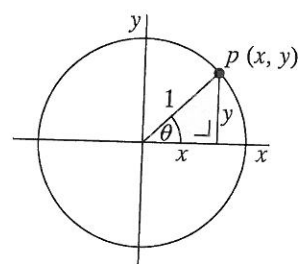
Take any point $p(x, y)$ on the circle, making an angle of θ , from the centre.

$$\cos \theta = \frac{x}{1} = x$$

$$\sin \theta = \frac{y}{1} = y$$

$$\tan \theta = \frac{y}{x} = \frac{\sin \theta}{\cos \theta}$$

This very important result indicates that the coordinates of any point on the unit circle can be represented by $p(\cos \theta, \sin \theta)$, where θ is any angle.



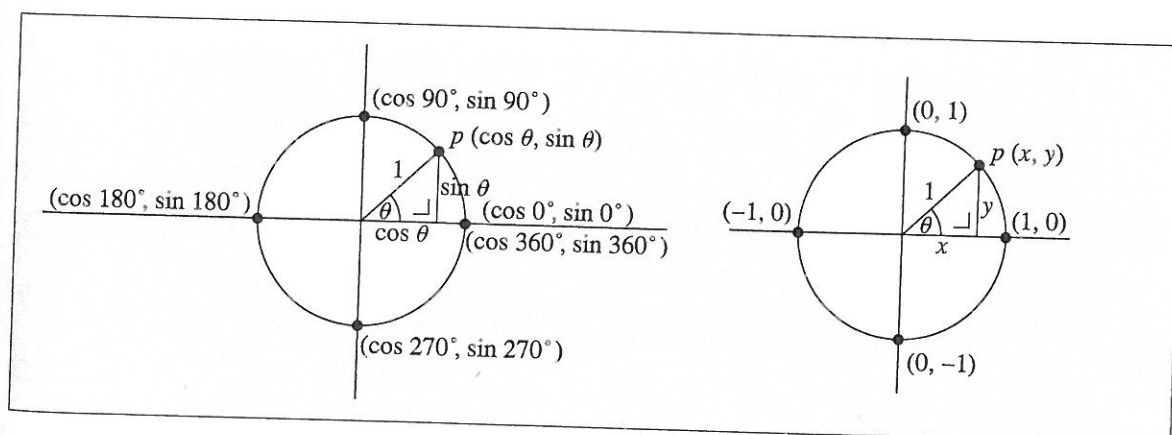
As the point p rotates, θ changes. These definitions of $\cos \theta$ and $\sin \theta$ in terms of the coordinates of a point rotating around the unit circle apply for **all** values of the angle θ° .

Memory Aid: (christian name, surname) = $(\cos \theta, \sin \theta) = (x, y)$

Note: Using Pythagoras's theorem: $\cos^2 \theta + \sin^2 \theta = 1$

Values of \sin , \cos and \tan for 0° , 90° , 180° , 270° and 360°

Both diagrams below represent the unit circle but using two different notations to describe any point p on the circle.



By comparing corresponding points on both unit circles, the values of \sin , \cos and \tan for 0° , 90° , 180° , 270° and 360° , can be read directly.

$(\cos 0^\circ, \sin 0^\circ) = (\cos 360^\circ, \sin 360^\circ) = (1, 0)$ $\cos 0^\circ = \cos 360^\circ = 1$ $\sin 0^\circ = \sin 360^\circ = 0$ $\tan 0^\circ = \tan 360^\circ = \frac{0}{1} = 0$	$(\cos 90^\circ, \sin 90^\circ) = (0, 1)$ $\cos 90^\circ = 0$ $\sin 90^\circ = 1$ $\tan 90^\circ = \frac{1}{0}$ (undefined)
$(\cos 180^\circ, \sin 180^\circ) = (-1, 0)$ $\cos 180^\circ = -1$ $\sin 180^\circ = 0$ $\tan 180^\circ = \frac{0}{-1} = 0$	$(\cos 270^\circ, \sin 270^\circ) = (0, -1)$ $\cos 270^\circ = 0$ $\sin 270^\circ = -1$ $\tan 270^\circ = \frac{-1}{0}$ (undefined)

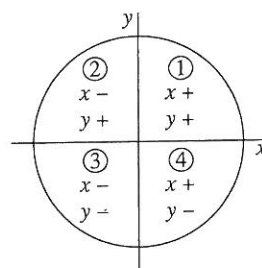
Note: Division by zero is undefined.

The x and y axes divide the plane into four quadrants. Consider the unit circle on the right:

$$\cos \theta = x \quad \sin \theta = y$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{y}{x}$$

By examining the signs of x and y in the four quadrants, the signs of $\sin \theta$, $\cos \theta$, and $\tan \theta$ for any value of θ can be found.



Summary of signs

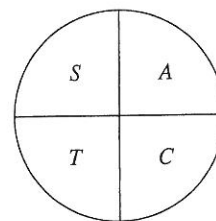
1st quadrant: \sin , \cos and \tan are all positive.

2nd quadrant: \sin is positive, \cos and \tan are negative.

3rd quadrant: \tan is positive, \sin and \cos are negative.

4th quadrant: \cos is positive, \sin and \tan are negative.

A very useful memory aid, *CAST*, in the diagram on the right, show the ratios that are positive for the angles between 0° and 360° .



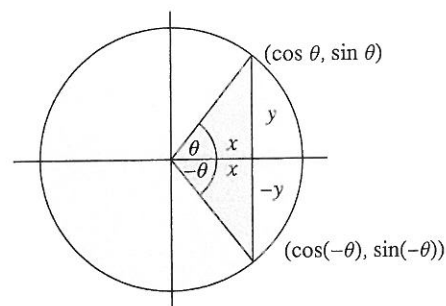
Negative angles

Consider the unit circle showing angles θ and $-\theta$.

$$\cos \theta = x \quad \sin \theta = y \quad \tan \theta = \frac{y}{x}$$


$$\cos(-\theta) = x \quad \sin(-\theta) = -y \quad \tan(-\theta) = -\frac{y}{x}$$

Thus,



$$\cos(-\theta) = \cos \theta \quad \sin(-\theta) = -\sin \theta \quad \tan(-\theta) = -\tan \theta$$

Method for finding the trigonometric ratio for any angle between 0° and 360° :

1. Draw a rough diagram of the angle.
2. Determine in which quadrant the angle lies and use  to find its sign.
3. Find its **related** angle (acute angle to nearest horizontal).
4. Use the trigonometric ratio of the related angle with the sign in step 2.

Example ▼

Find $\cos 210^\circ$, leaving your answer in surd form.

Solution:

Surd form, \therefore cannot use calculator.

1. The diagram shows the angle 210° .

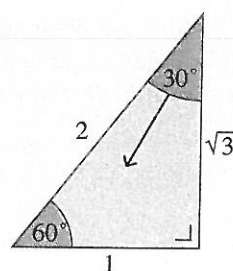
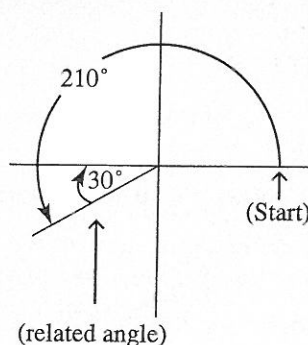
2. 210° is in the 3rd quadrant.
cos is negative in the 3rd quadrant.

3. Related angle is 30° .

$$4. \therefore \cos 210^\circ = -\cos 30^\circ$$

$$= -\frac{\sqrt{3}}{2}$$

(or use tables page 9)



Note: $\sin^2 A = (\sin A)^2$, $\cot^2 A = (\cot A)^2$, etc.

Exercise 7.2 ▼

Evaluate each of the following, answering in-surd form where necessary:

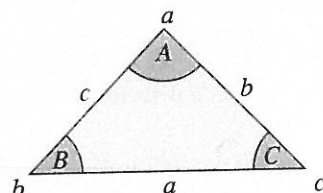
1. $\cos^2 45^\circ + \sin 30^\circ$
2. $1 - \cos^2 \frac{\pi}{6}$
3. $\sin^2 60^\circ + \cos^2 45^\circ$
4. $\tan^2 30^\circ \sin^2 60^\circ$
5. $\tan \frac{4\pi}{3} \tan \frac{\pi}{6}$
6. $\sin \frac{5\pi}{4} \cos \frac{3\pi}{4}$
7. $\frac{\sin^2 270^\circ + \cos^2 180^\circ}{2 \cos 0^\circ}$
8. $\frac{1 + \tan 60^\circ \tan 30^\circ}{\cos^2 45^\circ}$
9. $\tan 315^\circ - \sin 330^\circ$
10. $\tan^2 225^\circ - 2 \cos 240^\circ$
11. $\sin 315^\circ$
12. $\tan 120^\circ$
13. $\cos 150^\circ$
14. $\sin 135^\circ$
15. $\cos 330^\circ$
16. $\tan 300^\circ$
17. $\cos \frac{3\pi}{4}$
18. $\cos (-120^\circ)$
19. $\sin \left(-\frac{7\pi}{6}\right)$
20. $\tan(-150^\circ)$
21. $\cot \frac{\pi}{4}$
22. $\sec \frac{7\pi}{6}$
23. $\operatorname{cosec} 330^\circ$
24. $\cot 150^\circ$
25. $\operatorname{cosec} 120^\circ$
26. (i) Find the value of A for which $\cos A = -1$, $0^\circ \leq A \leq 360^\circ$.
 (ii) If $0^\circ \leq A \leq 360^\circ$, find the value of A for which $\sin A = 1$.
 (iii) If $0^\circ \leq A \leq 360^\circ$, find the values of A for which $\cos A = 0$.

Solution of Triangles

Notation

The diagram shows the **usual notation** for a triangle in trigonometry:

Vertices: a, b, c
Angles: A, B, C
Length of sides: a, b, c



The lengths of the sides are denoted by a lower case letter, and named after the angle they are opposite, i.e., a is opposite angle A , b is opposite angle B , and c is opposite angle C .

Using the same terminology we also have the following:

$$A = |\angle bac|,$$

$$B = |\angle abc|,$$

$$C = |\angle acb|$$

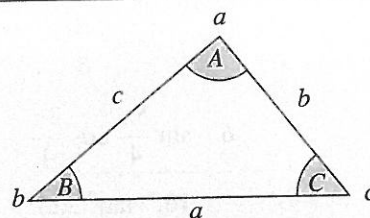
$$a = |bc|,$$

$$b = |ac|,$$

$$c = |ab|$$

Sine and Cosine Rule, Area of a Triangle

Sine Rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
 or $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$



Cosine Rule:

$$a^2 = b^2 + c^2 - 2bc \cos A \quad b^2 = a^2 + c^2 - 2ac \cos B \quad c^2 = a^2 + b^2 - 2ab \cos C$$

or

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad \cos B = \frac{a^2 + c^2 - b^2}{2ac} \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\text{Area of } \triangle abc = \frac{1}{2}ab \sin C = \frac{1}{2}ac \sin B = \frac{1}{2}bc \sin A$$

Use the sine rule if you know:

1. Two angles and one side.
2. Two sides and an angle opposite one of these sides.

Use the cosine rule if you know:

1. Two sides and the included angle.
2. The lengths of the three sides.

Notes: As a general rule, if you cannot use the sine rule then use the cosine rule.

If two angles are given we can work out the third angle straight away, as the three angles in a triangle add up to 180° .

The sine and cosine rules and the area of a triangle formulas also apply to a right-angled triangle, but with right-angled triangles we usually use the basic trigonometric definitions.

The largest angle of a triangle is opposite the largest side and the smallest angle is opposite the shortest side. There can be only one obtuse angle in a triangle.

Tackling problems in Trigonometry:

1. If not given, draw a diagram, and put in as much information as possible.
2. If two, or more, triangles are linked redraw the triangles separately.
3. Watch for common sides which link the triangles
(i.e. we can carry common values from one triangle to another triangle).
4. Use the sine or cosine rule as needed.

Example ▼

In $\triangle abc$, $|ab| = 3$, $|ac| = 5$ and $|bc| = 7$.

Calculate:

- (i) the measure of the greatest angle of the triangle
- (ii) the area of $\triangle abc$, giving your answer in the form $\frac{a\sqrt{b}}{c}$, where b is prime.

Solution:

- (i) The largest angle is opposite the largest side. Using the cosine rule:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$7^2 = 5^2 + 3^2 - 2(5)(3) \cos A$$

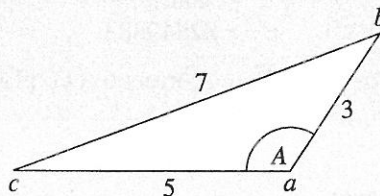
$$49 = 25 + 9 - 30 \cos A$$

$$30 \cos A = 25 + 9 - 49$$

$$30 \cos A = -15$$

$$\cos A = -\frac{1}{2}$$

$$A = \cos^{-1}\left(-\frac{1}{2}\right) = 120^\circ$$

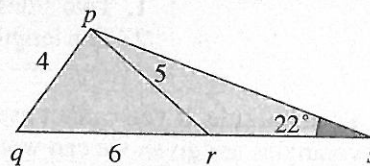


- (ii) Area of $\triangle abc = \frac{1}{2}bc \sin A = \frac{1}{2}(5)(3)\sin 120^\circ = \frac{1}{2}(5)(3)\left(\frac{\sqrt{3}}{2}\right) = \frac{15\sqrt{3}}{4}$

Example

In the diagram, $|pq| = 4$ cm, $|pr| = 5$ cm,
 $|qr| = 6$ cm and $|\angle psr| = 22^\circ$.

Find $|ps|$, correct to two places of decimals.



Solution:

Two triangles are linked and we need to work on them separately to find $|ps|$.

1. Consider Δpqr :

We need to use the cosine rule to find $|\angle prq|$, as we are given three sides.

$$\cos R = \frac{p^2 + q^2 - r^2}{2pq} \quad (\text{cosine rule})$$

$$\cos R = \frac{6^2 + 5^2 - 4^2}{2(6)(5)}$$

$$\cos R = \frac{45}{60} = \frac{3}{4}$$

$$R = \cos^{-1} \frac{3}{4} = 41.41^\circ$$

(correct to two places of decimals)

$$\therefore |\angle prs| = 180^\circ - 41.41^\circ = 138.59^\circ$$

2. Consider Δprs :

We now use the sine rule to find $|ps|$,
 (from diagram $|ps| = r$), as we have two angles and one side.

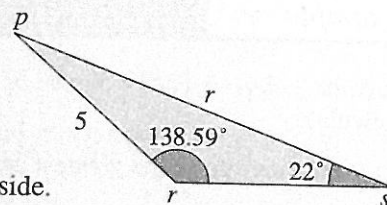
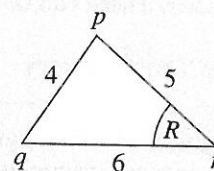
$$\frac{r}{\sin R} = \frac{s}{\sin S}$$

$$\frac{r}{\sin 138.59^\circ} = \frac{5}{\sin 22^\circ}$$

$$r = \frac{5 \sin 138.59^\circ}{\sin 22^\circ}$$

$$r = 8.82849883$$

Thus $|ps| = 8.83$ cm (correct to two places of decimals).



(sine rule, r missing so put that first)

(put in known values)

(multiply both sides by $\sin 138.59^\circ$)

Ambiguous case:

If $\sin A = \frac{\sqrt{3}}{2}$, then $A = 60^\circ$ or 120° : two possible solutions. Thus, we have to be very careful when using the sine rule to calculate an angle, as there may be two possible answers. Whenever we are given two sides and a non-included angle, there is a risk of having two triangles that satisfy the given conditions. However, it should not be assumed that there will always be two triangles satisfying the given conditions. For example, we might find that if we use the larger angle the sum of the three angles in the triangle is greater than 180° . The ambiguous case arises only when the smaller of the two given sides is opposite the known angle.

Example ▼

In a triangle, $a = 8$, $b = 9$ and $A = 60^\circ$. Find the possible values of B and C , correct to the nearest degree, and sketch the triangles.

Solution:

$$\begin{aligned}\frac{\sin B}{b} &= \frac{\sin A}{a} \\ \frac{\sin B}{9} &= \frac{\sin 60^\circ}{8} \\ \sin B &= \frac{9 \sin 60^\circ}{8} \\ B &= \sin^{-1}\left(\frac{9 \sin 60^\circ}{8}\right) \\ B &= 77^\circ\end{aligned}$$

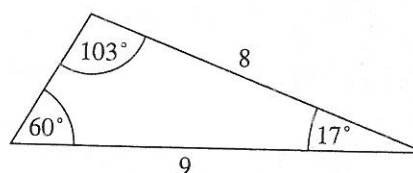
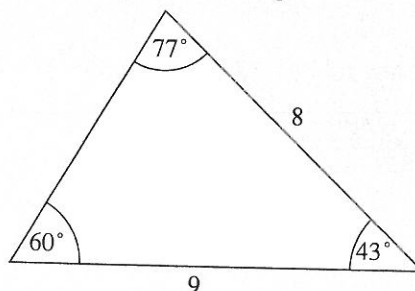
(correct to the nearest degree)

$$\text{or } B = 180^\circ - 77^\circ = 103^\circ$$

$$A = 60^\circ, B = 77^\circ, C = 180^\circ - 60^\circ - 77^\circ = 43^\circ$$

$$A = 60^\circ, B = 103^\circ, C = 180^\circ - 60^\circ - 103^\circ = 17^\circ$$

Two possible triangles



There is no ambiguity when using the cosine rule. This is because cosine is positive in the first quadrant (giving an acute angle), but negative in the second quadrant (giving an obtuse angle).

Example ▼

A vertical flagpole stands on horizontal ground. The angle of elevation of the top of the pole from a certain point on the ground is θ . From a point on the ground 10 metres closer to the pole the angle of elevation is β . Show that the height of the pole is

$$\frac{10 \sin \theta \sin \beta}{\sin(\beta - \theta)}$$

Solution:

Represent the situation with a diagram, then redraw the two triangles separately.
Put in known length.

Let x m = the length of the common side of both triangles.

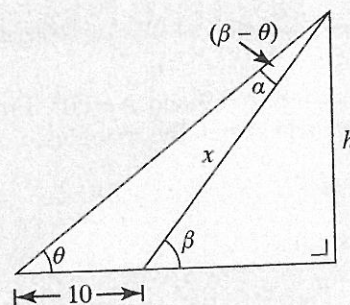
$$\alpha + \theta = \beta \quad (\text{exterior angle of a triangle})$$

$$\alpha = (\beta - \theta)$$

From the right-angled triangle,

$$\sin \beta = \frac{h}{x} \quad (h = \text{height of flagpole})$$

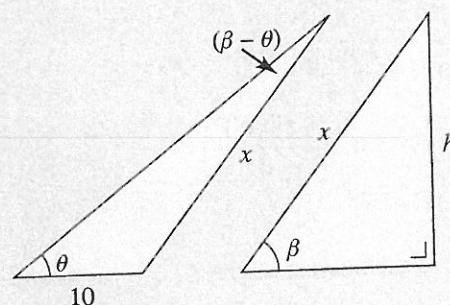
$$x = \frac{h}{\sin \beta} \quad ①$$



Using the sine rule on the other triangle,

$$\frac{x}{\sin \theta} = \frac{10}{\sin(\beta - \theta)}$$

$$x = \frac{10 \sin \theta}{\sin(\beta - \theta)} \quad ②$$



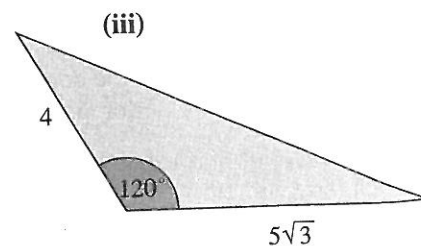
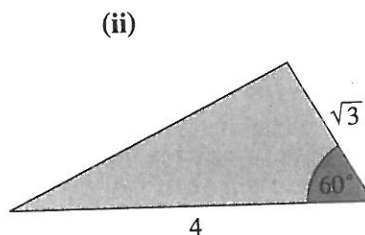
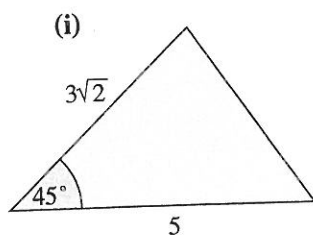
We now equate the two different expressions for x , the common side of both triangles.

$$\begin{aligned} x &= x \\ \frac{h}{\sin \beta} &= \frac{10 \sin \theta}{\sin(\beta - \theta)} \\ h &= \frac{10 \sin \theta \sin \beta}{\sin(\beta - \theta)} \end{aligned}$$

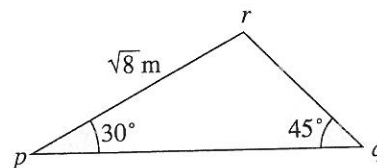
Exercise 7.3

Unless otherwise stated, where necessary give the lengths of sides and areas correct to two decimal places and give angles correct to one place of decimals.

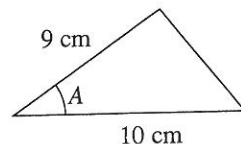
- In Δpqr , $p = 7$ cm, $P = 30^\circ$, $Q = 84^\circ$. Find R , q and r .
- In Δabc , $b = 8$ cm, $c = 10$ cm, $A = 60^\circ$. Find a .
- Find the area of these triangles, without using a calculator.



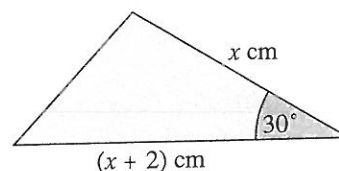
4. In Δpqr , $|pr| = \sqrt{8}$ m, $|\angle rpq| = 30^\circ$ and $|\angle pqr| = 45^\circ$. Show that the area of $\Delta pqr = 2.7$ m², correct to one place of decimals.



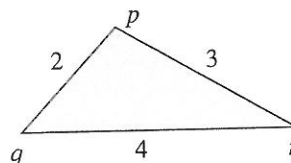
5. In the given triangle, $\cos A = \frac{4}{5}$. Without using tables or a calculator, find the area of the triangle.



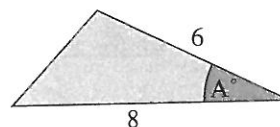
6. The area of the triangle shown on the right is 6 cm². Find the value of x .



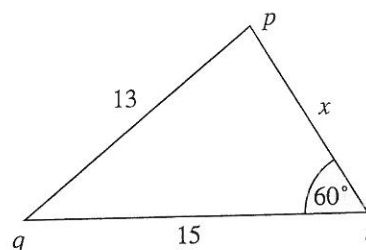
7. Calculate the three angles in triangle pqr , correct to two places of decimals.



8. The area of the triangle shown is 12 square units. Find two different values of A , and make a sketch of both triangles.

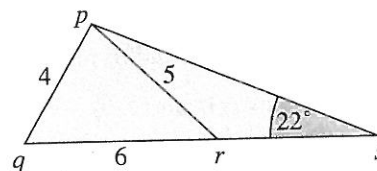


9. In triangle pqr , $|pq| = 13$, $|qr| = 15$ and $|pr| = x$. Find two possible values for x .

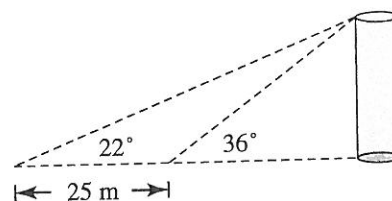


10. In triangle abc , $|ab| = 3\sqrt{3}$, $|bc| = 3\sqrt{2}$ and $|\angle bac| = \frac{\pi}{4}$. Find two possible values for $|\angle acb|$ and the corresponding two possible values of $|\angle abc|$, giving your answers in radians.

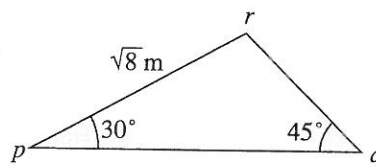
11. In the diagram, $|pq| = 4$ cm, $|pr| = 5$ cm, $|qr| = 6$ cm and $|\angle psr| = 22^\circ$. Find $|ps|$, correct to one place of decimals.



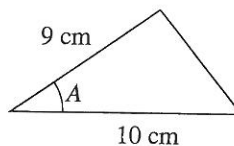
12. A surveyor wishes to measure the height of a round tower. Measuring the angle of elevation, he finds that the angle increases from 22° to 36° after walking 25 m towards the base of the tower. Calculate the height of the tower, correct to the nearest m.



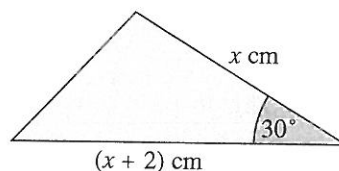
4. In Δpqr , $|pr| = \sqrt{8}$ m, $|\angle rpq| = 30^\circ$ and $|\angle pqr| = 45^\circ$. Show that the area of $\Delta pqr = 2.7$ m², correct to one place of decimals.



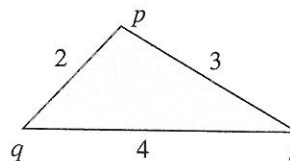
5. In the given triangle, $\cos A = \frac{4}{5}$. Without using tables or a calculator, find the area of the triangle.



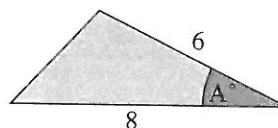
6. The area of the triangle shown on the right is 6 cm². Find the value of x .



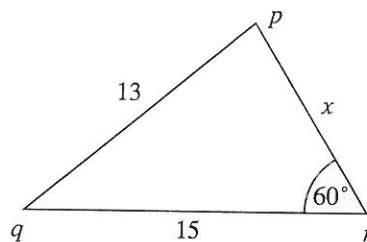
7. Calculate the three angles in triangle pqr , correct to two places of decimals.



8. The area of the triangle shown is 12 square units. Find two different values of A , and make a sketch of both triangles.

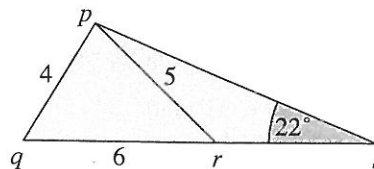


9. In triangle pqr , $|pq| = 13$, $|qr| = 15$ and $|pr| = x$. Find two possible values for x .

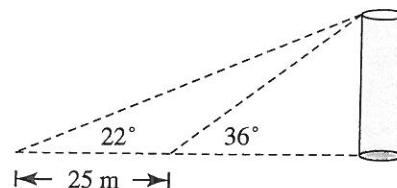


10. In triangle abc , $|ab| = 3\sqrt{3}$, $|bc| = 3\sqrt{2}$ and $|\angle bac| = \frac{\pi}{4}$. Find two possible values for $|\angle acb|$ and the corresponding two possible values of $|\angle abc|$, giving your answers in radians.

11. In the diagram, $|pq| = 4$ cm, $|pr| = 5$ cm, $|qr| = 6$ cm and $|\angle psr| = 22^\circ$. Find $|ps|$, correct to one place of decimals.



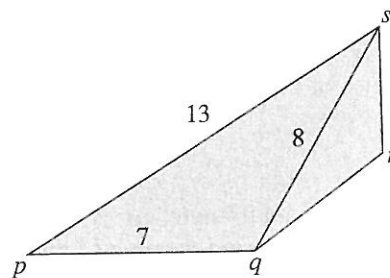
12. A surveyor wishes to measure the height of a round tower. Measuring the angle of elevation, he finds that the angle increases from 22° to 36° after walking 25 m towards the base of the tower. Calculate the height of the tower, correct to the nearest m.



13. $pqrs$ is a quadrilateral. $|pq| = 7$, $|qs| = 8$, $|ps| = 13$.

(i) Show $|\angle pqs| = 120^\circ$.

(ii) Given that the quadrilateral $pqrs$ has area $\frac{35\sqrt{3}}{2}$ find the ratio, area Δpqs : area Δqrs .



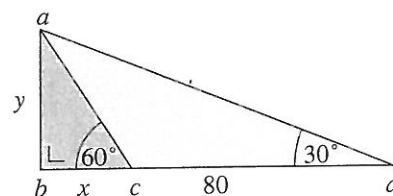
14. In triangle pqr , $|pr| = 5$, $|qr| = 4$ and $|\angle qpr| = 37^\circ$. Find two possible values of $|\angle pqr|$, correct to the nearest degree.

15. In the diagram, $ab \perp bd$, $|ab| = y$, $|bc| = x$ and $|cd| = 80$.

(i) Using Δabc , express x in terms of y

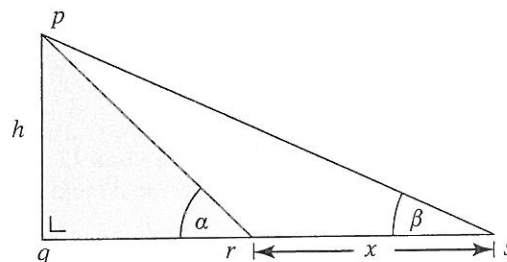
(ii) Using Δabd , express x in terms of y .

Hence, or otherwise, find y , expressing your answer in the form $a\sqrt{b}$, where b is prime.



16. In the given diagram, $pq \perp qs$, $|pq| = h$ and $|rs| = x$.

Show that $x = \frac{h \sin(\alpha - \beta)}{\sin \alpha \sin \beta}$.



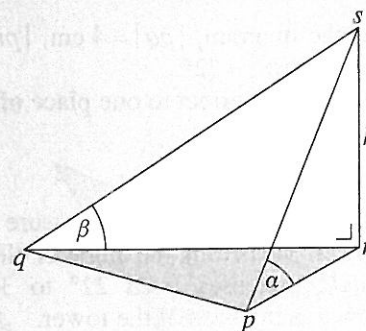
Three-dimensional Problems

When tackling problems in three dimensions it is good practice to redraw each triangle separately and apply the sine and cosine rule to these triangles. Watch for common sides which link the triangles. We can carry common values from one triangle to another triangle.

Example ▼

p , q and r are points on level ground. $[sr]$ is a vertical tower of height h . The angles of elevation of the top of the tower from p and q are α and β , respectively.

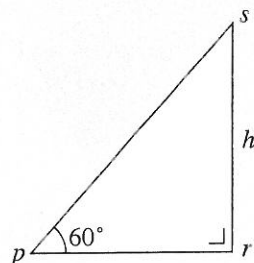
- (i) If $|\alpha| = 60^\circ$ and $|\beta| = 30^\circ$, express $|pr|$ and $|qr|$ in terms of h .
- (ii) Find $|qp|$ in terms of h , if $\tan \angle qrp = \sqrt{8}$.



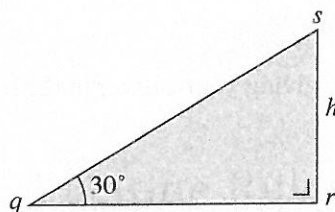
Solution:

Redraw right-angled triangles prs and qrs separately.

(i)



$$\begin{aligned}\tan 60^\circ &= \frac{h}{|pr|} \\ \sqrt{3} &= \frac{h}{|pr|} \\ \sqrt{3}|pr| &= h \\ |pr| &= \frac{h}{\sqrt{3}}\end{aligned}$$



$$\begin{aligned}\tan 30^\circ &= \frac{h}{|qr|} \\ \frac{1}{\sqrt{3}} &= \frac{h}{|qr|} \\ |qr| &= \sqrt{3}h\end{aligned}$$

(ii) Redraw triangle qpr separately.

Using the cosine rule on triangle qpr :

$$|qp|^2 = |qr|^2 + |pr|^2 - 2|qr| \cdot |pr| \cos \angle qrp$$

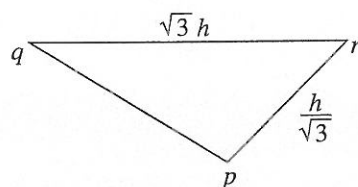
$$|qp|^2 = (\sqrt{3}h)^2 + \left(\frac{h}{\sqrt{3}}\right)^2 - 2(\sqrt{3}h)\left(\frac{h}{\sqrt{3}}\right)\left(\frac{1}{3}\right)$$

$$|qp|^2 = 3h^2 + \frac{h^2}{3} - \frac{2}{3}h^2$$

$$|qp|^2 = \frac{9h^2 + h^2 - 2h^2}{3}$$

$$|qp|^2 = \frac{8h^2}{3}$$

$$|qp| = \sqrt{\frac{8h^2}{3}} = \sqrt{\frac{8}{3}}h$$

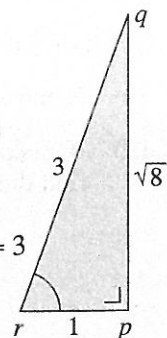


Draw a right angle to get $\cos \angle qrp$.

Given: $\tan \angle qrp = \sqrt{8} = \frac{\sqrt{8}}{1}$

Using Pythagoras's theorem, $|qr| = 3$

$$\therefore \cos \angle qrp = \frac{1}{3}$$



Exercise 7.4

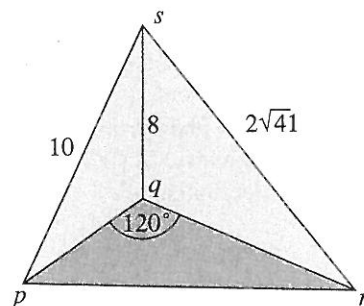
1. p , q and r are points on horizontal ground.

$[qs]$ represents a vertical pole of height 8 m.

If $\angle pqr = 120^\circ$, $|ps| = 10$ m and $|rs| = 2\sqrt{41}$ m, find:

(i) $|pq|$ (ii) $|qr|$ (iii) $|pr|$

Calculate the area of triangle pqr . Express your answer in the form $a\sqrt{b}$ m², where b is prime.

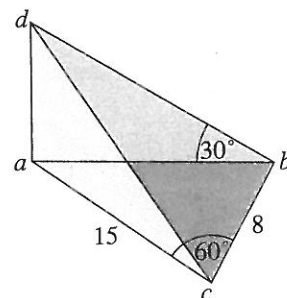


2. Points a, b and c are on horizontal ground.
 $[ad]$ represents a vertical pole.

$$|ac| = 15 \text{ m}, |bc| = 8 \text{ m}, \angle acb = 60^\circ \text{ and } \angle abd = 30^\circ.$$

Calculate:

- (i) $|ab|$
 (ii) $|ad|$, giving your answer in the form $\frac{a\sqrt{b}}{b}$.

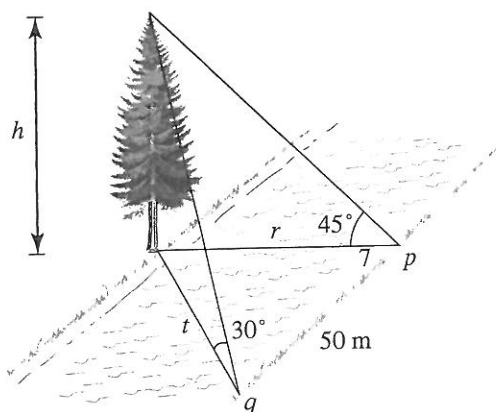


3. The diagram shows a river with parallel banks, r metres apart.

A tree, of height h metres, is directly opposite from a point p , as shown.

A woman wants to measure the height of the tree. From p the angle of elevation of the top of the tree is 45° . She then walks to a point q , 50 metres downstream, so that the horizontal distance from q to the base of the tree is t metres. From q the angle of elevation of the top of the tree is 30° .

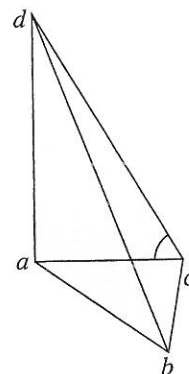
- (i) Express r and t in terms of h .
 (ii) Write down an equation involving r and t .
 (iii) Hence, calculate (a) h (b) t , correct to one place of decimals.



4. a, b and c are points on level ground.
 $[ad]$ represents a vertical pole.

$$|ab| = x, |ac| = \frac{\sqrt{3}}{2}x \text{ and } \angle abd = \frac{\pi}{3}.$$

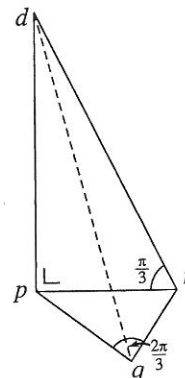
- (i) Express $|ad|$ in terms of x .
 (ii) If $\angle acd = \tan^{-1} k$, $k \in \mathbb{N}$, find the value of k .



5. Points p, q, r are on the horizontal.

$$|pq| = 5, |qr| = 3 \text{ and } \angle pqr = \frac{2\pi}{3}.$$

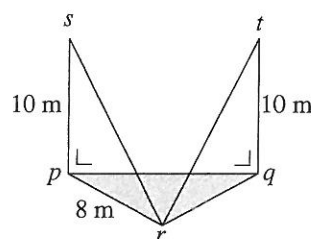
- (i) Calculate $|pr|$.
 (ii) $[pd]$ represents a vertical mast. The angle of elevation of d from r is $\frac{\pi}{3}$. Find $|dq|$, giving your answer in the form $2\sqrt{a}$ and calculate the measure of $\angle pqr$, correct to one place of decimals.



6. $[sp]$, $[tq]$ are vertical poles each of height 10 m, p , q , r are points on level ground. Two wires of equal length join s and t to r , i.e. $|sr| = |tr|$.

If $|pr| = 8$ m and $|\angle prq| = 120^\circ$, calculate:

- $|pr|$ in the form $a\sqrt{b}$, where b is prime.
- $|sr|$ in the form \sqrt{c} .
- $|\angle srt|$ to the nearest degree.



Identities involving the Sine and Cosine Rules

We often have to prove identities involving the usual notation for a triangle using the sine and cosine rules. This usually involves rearranging the sine or cosine rule and substituting the rearranged expression to prove the required identity.

Cosine rule

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\text{then } \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Similarly for B and C .

Sine rule

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\text{then } \sin A = \frac{a \sin B}{b}$$

$$\text{or } \sin B = \frac{b \sin A}{a}$$

Example

Using the usual notation for a triangle, prove that $c(b \cos A - a \cos B) = b^2 - a^2$.

Solution:

From the cosine rule,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad \text{and} \quad \cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\begin{aligned} c(b \cos A - a \cos B) &= bc \cos A - ac \cos B \\ &= bc \left(\frac{b^2 + c^2 - a^2}{2bc} \right) - ac \left(\frac{a^2 + c^2 - b^2}{2ac} \right) \end{aligned}$$

[this is the substitution]

$$\begin{aligned} &= \frac{(b^2 + c^2 - a^2)}{2} - \frac{(a^2 + c^2 - b^2)}{2} \\ &= \frac{(b^2 + c^2 - a^2) - (a^2 + c^2 - b^2)}{2} \end{aligned}$$

[same denominator]

$$\begin{aligned} &= \frac{b^2 + c^2 - a^2 - a^2 - c^2 + b^2}{2} \\ &= \frac{2b^2 - 2a^2}{2} \\ &= b^2 - a^2 \end{aligned}$$

Exercise 7.5 ▼

Prove each of the following identities using the usual notation for a triangle:

1. $b \sin C = c \sin B$

2. $b \cos C + c \cos B = a$

3. $bc \cos A + ac \cos B = c^2$

4. $a(b \cos C - c \cos B) = b^2 - c^2$

5. $\frac{1}{c \cos B - b \cos C} = \frac{a}{c^2 - b^2}$

6. $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}$

7. $a(\sin B - \sin C) + b(\sin C - \sin A) + c(\sin A - \sin B) = 0$

8. $\tan A = \frac{a \sin C}{b - a \cos C}$

9. $\frac{b - c}{b + c} = \frac{\sin B - \sin C}{\sin B + \sin C}$

10. (i) $c = a \cos B + b \cos A$.

(ii) $a + b + c = (b + c) \cos A + (a + c) \cos B + (a + b) \cos C$

11. $\frac{\cos B}{b} - \frac{\cos C}{c} = \frac{c^2 - b^2}{abc}$

12. $a^2 + b^2 + c^2 = 2(bc \cos A + ac \cos B + ab \cos C)$

13. What can you deduce about angle A in triangle abc , using the usual notation? – if:

(i) $a^2 > b^2 + c^2$

(ii) $a^2 = b^2 + c^2$

(iii) $a^2 < b^2 + c^2$

(iv) $a^2 = b^2 + c^2 - bc$

(v) $a^2 = b^2 + c^2 + bc$

(vi) $a^2 = b^2 + c^2 - \sqrt{3}bc$

If $A = \frac{\pi}{4}$ and $B = \frac{\pi}{3}$, show that $3a^2 = 2b^2$.

Proving Trigonometric Identities

An identity is an equation that is true for all values of the variable.
Some identities we have met so far are:

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\cot \theta = \frac{1}{\tan \theta} = \frac{\cos \theta}{\sin \theta}$$

$$\sec \theta = \frac{1}{\cos \theta}$$

$$\operatorname{cosec} \theta = \frac{1}{\sin \theta}$$

$$\cos(-\theta) = \cos \theta \quad \sin(-\theta) = -\sin \theta \quad \tan(-\theta) = -\tan \theta$$

Method for proving trigonometric identities:

The method is to take one side and convert it into the other side. It is usually easier to start with the side which is more complicated.

Example ▼

Prove that $\cos^2 \theta + \sin^2 \theta = 1$.

Hence, prove (i) $1 + \tan^2 \theta = \sec^2 \theta$ (ii) $\cot^2 \theta + 1 = \operatorname{cosec}^2 \theta$.

Solution:

Let θ be any angle as shown in diagram.

The coordinates of p are (x, y) and $|op| = r$.

$$\cos \theta = \frac{x}{r} \Rightarrow \cos^2 \theta = \frac{x^2}{r^2}$$

$$\sin \theta = \frac{y}{r} \Rightarrow \sin^2 \theta = \frac{y^2}{r^2}$$

$$x^2 + y^2 = r^2$$

(Pythagoras' Theorem)

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

(divide both sides by r^2)

$$\cos^2 \theta + \sin^2 \theta = 1$$

$$(i) \quad \cos^2 \theta + \sin^2 \theta = 1$$

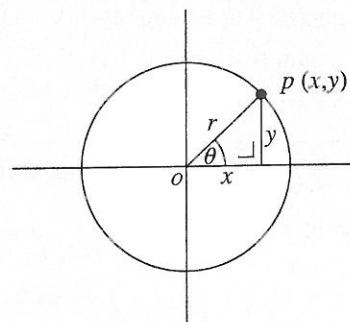
$$\frac{\cos^2 \theta}{\cos^2 \theta} + \frac{\sin^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}$$

$$\begin{aligned} & \text{(dividing both sides by } \cos^2 \theta) \\ & 1 + \tan^2 \theta = \sec^2 \theta \end{aligned}$$

$$(ii) \quad \cos^2 \theta + \sin^2 \theta = 1$$

$$\frac{\cos^2 \theta}{\sin^2 \theta} + \frac{\sin^2 \theta}{\sin^2 \theta} = \frac{1}{\sin^2 \theta}$$

$$\begin{aligned} & \text{(dividing both sides by } \sin^2 \theta) \\ & \cot^2 \theta + 1 = \operatorname{cosec}^2 \theta \end{aligned}$$



Example ▼

Prove:

$$(i) \quad \frac{1}{1 - \sin A} + \frac{1}{1 + \sin A} = 2 \sec^2 A$$

$$(ii) \quad \cot \theta + \frac{\sin \theta}{1 + \cos \theta} = \operatorname{cosec} \theta.$$

Solution:

$$\begin{aligned} (i) \quad & \frac{1}{1 - \sin A} + \frac{1}{1 + \sin A} \\ &= \frac{1(1 + \sin A) + 1(1 - \sin A)}{(1 - \sin A)(1 + \sin A)} \\ &= \frac{1 + \sin A + 1 - \sin A}{1 + \sin A - \sin A - \sin^2 A} \\ &= \frac{2}{1 - \sin^2 A} \\ &= \frac{2}{\cos^2 A} \\ &= 2 \frac{1}{\cos^2 A} \\ &= 2 \sec^2 A \end{aligned}$$

$$\begin{aligned} (ii) \quad & \cot \theta + \frac{\sin \theta}{1 + \cos \theta} \\ &= \frac{\cos \theta}{\sin \theta} + \frac{\sin \theta}{1 + \cos \theta} \\ &= \frac{\cos \theta(1 + \cos \theta) + \sin \theta(\sin \theta)}{\sin \theta(1 + \cos \theta)} \\ &= \frac{\cos \theta + \cos^2 \theta + \sin^2 \theta}{\sin \theta(1 + \cos \theta)} \\ &= \frac{(1 + \cos \theta)}{\sin \theta(1 + \cos \theta)} \\ &= \frac{1}{\sin \theta} \\ &= \operatorname{cosec} \theta \end{aligned}$$

Exercise 7.6 ▼

Prove each of the following identities:

1. $\sec A \sin A = \tan A$
3. $\tan \theta \sqrt{1 - \sin^2 \theta} = \sin \theta$
5. $\operatorname{cosec}^2 A = 1 + \cot^2 A$
7. $\frac{\tan \theta}{\sqrt{1 + \tan^2 \theta}} = \sin \theta$
9. $\frac{1 + \tan^2 A}{\sec^3 A} = \cos^2 A$
11. $(\sin A + \cos A)^2 + (\sin A - \cos A)^2 = 2$
13. $\frac{1}{1 + \cos A} + \frac{1}{1 - \cos A} = 2 \operatorname{cosec}^2 A$
15. $(\sin \theta + \cos \theta)(1 - \sin \theta \cos \theta) = \sin^3 \theta + \cos^3 \theta$
16. $(\sec A + \tan A - 1)(\sec A - \tan A + 1) = 2 \tan A$
17. $(1 + \tan \theta)^2 + (1 - \tan \theta)^2 = 2 \sec^2 \theta$
19. $\frac{1 - \cos \theta}{\sin \theta} = \frac{1}{\operatorname{cosec} \theta + \cot \theta}$
21. $\frac{\tan \theta + \cot \theta}{\sec \theta + \operatorname{cosec} \theta} = \frac{1}{\sin \theta + \cos \theta}$
2. $(1 - \cos A)(1 + \cos A) = \sin^2 A$
4. $(1 + \tan^2 \theta)(1 - \sin^2 \theta) = 1$
6. $\sec^2 \theta - \tan^2 \theta = 1$
8. $\frac{\cos A}{1 - \sin A} - \tan A = \sec A$
10. $\frac{1}{1 + \tan^2 A} + \frac{1}{1 + \cot^2 A} = 1$
12. $\sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$
14. $\frac{\cos \theta}{1 + \sin \theta} + \frac{1 + \sin \theta}{\cos \theta} = 2 \sec \theta$
18. $\sec \theta + \operatorname{cosec} \theta \cot \theta = \sec \theta \operatorname{cosec}^2 \theta$
20. $(\tan \theta + \sec \theta)^2 = \frac{1 + \sin \theta}{1 - \sin \theta}$
22. $\frac{\sin A}{\sqrt{1 + \cot^2 A}} + \frac{\cos A}{\sqrt{1 + \tan^2 A}} = 1$