CHAPTER 2

QUADRATIC AND CUBIC EQUATIONS

Quadratic Equations

Any equation of the form $ax^2 + bx + c = 0$, $a \neq 0$, is called a quadratic equation.
To solve a quadratic equation we either:

> **1.** Factorise and let each factor = 0; or
>
> **2.** Use the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Example

(i) Solve $6x^2 - 11x - 10 = 0$.
(ii) Solve $x^2 + 4x - 1 = 0$, giving your solutions in surd form.

Solution:

(i)

$$6x^2 - 11x - 10 = 0$$

$$(3x + 2)(2x - 5) = 0 \qquad \text{(factorise the left-hand side)}$$

$$3x + 2 = 0 \quad \text{or} \quad 2x - 5 = 0 \qquad \text{(let each factor = 0)}$$

$$3x = -2 \quad \text{or} \quad 2x = 5$$

$$x = -\tfrac{2}{3} \quad \text{or} \quad x = \tfrac{5}{2} \qquad \text{(solve each simple equation)}$$

(ii) $x^2 + 4x - 1 = 0$ (answers in surd form $\therefore$ use formula)

$$\boxed{\begin{array}{c} x^2 + 4x - 1 = 0 \\ a = 1, b = 4, c = -1 \end{array}}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(-1)}}{2(1)}$$

$$x = \frac{-4 \pm \sqrt{16 + 4}}{2}$$

$$x = \frac{-4 \pm \sqrt{20}}{2}$$

$$x = \frac{-4 \pm 2\sqrt{5}}{2}$$

$$x = -2 \pm \sqrt{5}$$

$$\therefore \; x = -2 + \sqrt{5} \quad \text{or} \quad x = -2 - \sqrt{5}.$$

$$\boxed{\begin{array}{l} \sqrt{20} \\ = \sqrt{4 \times 5} \\ = \sqrt{4}\sqrt{5} \\ = 2\sqrt{5} \end{array}}$$

In some questions we can use the roots of one quadratic equation to help us to solve another quadrat equation by using a substitution.

Example

Solve the equation $x - 11 + \frac{24}{x} = 0$.

Hence, solve $(y^2 - 2y) - 11 + \frac{24}{(y^2 - 2y)} = 0$.

Solution:

$$x - 11 + \frac{24}{x} = 0$$

$x^2 - 11x + 24 = 0$ (multiply each part by x)

$(x - 3)(x - 8) = 0$ (factorise the left-hand side)

$x - 3 = 0$ or $x - 8 = 0$ (let each factor $= 0$)

$x = 3$ or $x = 8$ (solve each simple equation)

let $(y^2 - 2y) = x$ (this is the substitution)

$y^2 - 2y = 3$	$y^2 - 2y = 8$
$y^2 - 2y - 3 = 0$	$y^2 - 2y - 8 = 0$
$(y + 1)(y - 3) = 0$	$(y + 2)(y - 4) = 0$
$y + 1 = 0$ or $y - 3 = 0$	$y + 2 = 0$ or $y - 4 = 0$
$y = -1$ or $y = 3$	$y = -2$ or $y = 4$

(or)

$\therefore \quad y = -2, -1, 3, 4$

Exercise 2.1

Solve each of the following equations:

1. $2x^2 + 5x - 12 = 0$
2. $x^2 - 3x = 0$
3. $x^2 - 4 = 0$
4. $3x^2 + 14x + 8 = 0$
5. $5x^2 + 14x - 3 = 0$
6. $x^2 - 6x + 9 = 0$
7. $2x^2 = 3x$
8. $6x^2 - x = 2$
9. $9x^2 - 12x + 4 = 0$
10. $8x^2 = 9 - 6x$
11. $15x^2 + x - 6 = 0$
12. $4x^2 - 25 = 0$

Solve each of the following equations, giving your solutions in surd form:

13. $x^2 + 6x + 4 = 0$
14. $x^2 - 4x + 1 = 0$
15. $x^2 - 8x + 13 = 0$
16. $x^2 - 2x - 2 = 0$
17. $x^2 - 4x - 14 = 0$
18. $x^2 + 10x - 23 = 0$

Write each of the following equations in the form $ax^2 + bx + c = 0$, and hence, solve each equation:

19. $1 - \frac{5}{x} + \frac{6}{x^2} = 0$
20. $3 + \frac{2}{x - 2} = \frac{1}{x}$
21. $\frac{1}{2} = \frac{1}{x - 1} - \frac{1}{x}$
22. $\frac{1}{2} - \frac{1}{x} = \frac{1}{x + 3}$
23. $\frac{5}{2} = \frac{1}{x - 1} + \frac{3}{x + 2}$
24. $\frac{2}{9} = \frac{1}{x - 5} - \frac{1}{x + 1}$

25. Solve $\frac{1}{4} - \frac{1}{x+2} = \frac{1}{x-2}$, giving your answers in the form $a \pm b\sqrt{c}$.

26. Solve $x^2 - x - 20 = 0$. Hence, solve $\left(2k + \frac{2}{k}\right)^2 - \left(2k + \frac{2}{k}\right) - 20 = 0$.

27. Solve $3x^2 + 16x - 12 = 0$. Hence, solve $3\left(y - \frac{7}{y}\right)^2 + 16\left(y - \frac{7}{y}\right) - 12 = 0$.

28. Solve $x^2 - 2x - 24 = 0$. Hence, solve $\left(x + \frac{4}{x}\right)^2 - 2\left(x + \frac{4}{x}\right) - 24 = 0$.

29. Solve $x^2 - 6x + 8 = 0$. Hence, solve $\left(x + \frac{1}{x}\right)^2 - 6\left(x + \frac{1}{x}\right) + 8 = 0$.

30. Solve: **(i)** $x^4 - 13x^2 + 36 = 0$ **(ii)** $x^4 - 17x^2 + 16 = 0$.

Modulus and Irrational Equations

Modulus Equations

The modulus of x, written $|x|$, is defined as its positive or absolute value.
For example, $|5| = 5$ and $|-2| = 2$.
A modulus equation is one where the variable is contained within a modulus.
For example, $|x - 1| = 4$, is a modulus equation.
Note: If $|x| = 3$, then $x = 3$ or $x = -3$.

Modulus equations are solved with the following steps:

1. Arrange to have the modulus part by itself on one side of the equation.
2. Square both sides (this removes the modulus bars).
3. Solve the resultant equation.

Note: If there are two modulus parts, arrange to have one modulus part on each side.

Example

Solve $2|x-2| - |x+3| = 0$

Solution:

$2|x-2| - |x+3| = 0$

$2|x-2| = |x+3|$ (one modulus on each side)

$(2|x-2|)^2 = (|x+3|)^2$ (square both sides)

$4(x^2 - 4x + 4) = x^2 + 6x + 9$ ($(ab)^2 = a^2b^2$)

$4x^2 - 16x + 16 = x^2 + 6x + 9$ (remove brackets)

$3x^2 - 22x + 7 = 0$

$(3x - 1)(x - 7) = 0$

$3x - 1 = 0$ or $x - 7 = 0$

$3x = 1$ or $x = 7$

$x = \frac{1}{3}$ or $x = 7$

Irrational Equations

An irrational equation is one where the variable is contained under a square root.
For example, $\sqrt{x+2} = x - 4$ is an irrational equation.

Irrational equations are solved with the following steps:

1. Arrange to have the surd (root) part on its own on one side.
2. Square both sides.
3. Solve the resultant equation.
4. Test every solution in the **original** equation.

Note: Sometimes after squaring both sides there will still be a surd part left in the equation. In t case, arrange to have this surd part on its own on one side and then square both sides again.

Example

Solve $x = \sqrt{19-2x} + 2$

Solution:

$x = \sqrt{19-2x} + 2$	
$(x-2) = (\sqrt{19-2x})$	(rearrange with surd part on its own)
$(x-2)^2 = (\sqrt{19-2x})^2$	(square both sides)
$x^2 - 4x + 4 = 19 - 2x$	(remove brackets)
$x^2 - 2x - 15 = 0$	(write in the form $ax^2 + bx + c = 0$)
$(x+3)(x-5) = 0$	(factorise left-hand side)
$x + 3 = 0$ or $x - 5 = 0$	(let each factor = 0)
$x = -3$ or $x = 5$	(solve each simple equation)

Check $x = -3$: $\quad -3 = \sqrt{19-2(-3)} + 2 = \sqrt{25} + 2 = 5 + 2 = 7$ **False**
Check $x = 5$: $\quad 5 = \sqrt{19-2(5)} + 2 = \sqrt{9} + 2 = 3 + 2 = 5$ **True**

$\therefore$ $x = 5$ is the only solution.

Note: Squaring both sides introduced a new root, called an extraneous root, $x = -3$.
This does not satisfy the original equation and hence is rejected.

Note: The square root of a number is defined as the 'positive square root'.
For example, $\sqrt{16} = 4$, not ± 4.

Exercise 2.2

Solve each of the following equations:

1. $|x-1| = 4$ **2.** $|x-2| = 3$ **3.** $|x+3| = 5$

4. $|2x-1| = 3$ **5.** $|3x-1| - 4 = 0$ **6.** $2|x-1| = 3$

7. $|x+1| = |x-2|$ **8.** $|2x+1| = |x-1|$ **9.** $|2x-1| - x =$

10. $|4-3x|-|2x-1|=0$

11. $|3x-1|-|1-2x|=0$

12. $2|x-1|=|x+1|$

13. $2|x+1|-|x+3|=0$

14. $3|x+1|=|2x-1|$

15. $|2-x|=\frac{1}{2}|x|$

16. $\left|\frac{x-2}{3}\right|=1$

17. $\left|\frac{3x+1}{x-1}\right|=2$

18. $\left|\frac{2x+1}{x+2}\right|=\frac{1}{2}$

19. $x=\sqrt{5x-4}$

20. $x=\sqrt{x+6}$

21. $x+6=5\sqrt{x}$

22. $x-2=\sqrt{2x-1}$

23. $2x-1=\sqrt{8x+1}$

24. $2x-7=\sqrt{x^2-3x-1}$

25. $x=\sqrt{3x-5}+1$

26. $x-\sqrt{x+3}=3$

27. $x+1=3\sqrt{x-1}$

Questions 28–33 require squaring twice:

28. $\sqrt{x}+1=\sqrt{x+9}$

29. $1+\sqrt{x}=\sqrt{3(x-1)}$

30. $2\sqrt{x}=\sqrt{4x-11}+1$

31. $\sqrt{3x}+1=\sqrt{5x+1}$

32. $\sqrt{x}+2=\sqrt{2x+7}$

33. $\sqrt{3x-2}=2+\sqrt{x-2}$

34. Solve **(i)** $\frac{|1-3x|}{\sqrt{x^2+1}}=\sqrt{8}$ **(ii)** $\frac{5|x-1|}{\sqrt{x^2+1}}=\sqrt{10}$

Simultaneous Equations, One Linear and One Quadratic

The **method of substitution** is used to solve between a linear equation and a quadratic equation.

The method involves three steps:

1. From the linear equation express one variable in terms of the other.
2. Substitute this into the quadratic equation and solve.
3. Substitute separately the value(s) obtained in step 2 into the linear equation in step 1 to find the corresponding value(s) of the other variable.

Example

Solve the simultaneous equations $2x-3y-1=0$ and $x^2+xy-4y^2=2$.

Solution:

1. $2x-3y-1=0$ (get x, or y, on its own from the linear equation)

$2x=3y+1$

$x=\left(\frac{3y+1}{2}\right)$ (x on its own)

2.

$$x^2+xy-4y^2=2$$

$$\left(\frac{3y+1}{2}\right)^2+\left(\frac{3y+1}{2}\right)y-4y^2=2 \qquad \left(\text{put in } \left(\frac{3y+1}{2}\right) \text{ for } x\right)$$

$$\frac{(9y^2+6y+1)}{4}+\frac{(3y^2+y)}{2}-4y^2=2$$

$$(9y^2+6y+1)+2(3y^2+y)-16y^2=8 \qquad \text{(multiply each part by 4)}$$
$$9y^2+6y+1+6y^2+2y-16y^2=8$$
$$-y^2+8y-7=0$$
$$y^2-8y+7=0$$
$$(y-1)(y-7)=0$$
$$y-1=0 \quad \text{or} \quad y-7=0$$
$$y=1 \quad \text{or} \quad y=7$$

3. Substitute separately $y=1$ and $y=7$ into the linear equation.

$y=1$: $\quad x=\frac{3y+1}{2}=\frac{3(1)+1}{2}=\frac{4}{2}=2$

$y=7$: $\quad x=\frac{3y+1}{2}=\frac{3(7)+1}{2}=\frac{22}{2}=11$

Thus, the solutions are $x=2$, $y=1$ or $x=11$, $y=7$.

Exercise 2.3

Solve the following pairs of simultaneous equations:

1. $x+2y=5$
$x^2+y^2=10$

2. $x-y=1$
$xy=42$

3. $3x-y-5=0$
$xy-x=0$

4. $x+y-6=0$
$x^2+2y^2-24=0$

5. $x-y-3=0$
$x^2-3y^2=13$

6. $x+y=8$
$x^2+xy+y^2=52$

7. $2x+y=3$
$x^2+xy+y^2=3$

8. $3x+y-5=0$
$2x^2+2xy+y^2=10$

9. $x+y=3$
$2x^2+3xy+2y^2=16$

10. $\frac{3x}{y}=1+\frac{7}{y}$
$x^2-xy+y^2=7$

11. $1=\frac{2}{x}-\frac{2y}{x}$
$x^2+2xy-8=0$

12. $\frac{x}{y}+1=\frac{10}{y}$
$x^2-y^2=40$

13. $2x-3y-1=0$
$x^2-2xy-3y^2+3=0$

14. $5x-2y+2=0$
$x^2+4y^2+x+2y-58=0$

15. $2x+3y+4=0$
$(x+3y)(2x-y)=4$

Sum and Product of the Roots of a Quadratic Equation

The quadratic equation $ax^2+bx+c=0$ can be written $x^2+\frac{b}{a}x+\frac{c}{a}=0$.

The roots of this equation are usually denoted by α and β.
Now we can write down a quadratic equation with roots α and β.

$$x=\alpha \quad \text{and} \quad x=\beta$$
$$x-\alpha=0 \quad \text{and} \quad x-\beta=0$$
$$(x-\alpha)(x-\beta)=0$$
$$x^2-\alpha x-\beta x+\alpha\beta=0$$
$$x^2-(\alpha+\beta)x+\alpha\beta=0$$

Thus, $\quad x^2-(\alpha+\beta)x+\alpha\beta=x^2+\frac{b}{a}x+\frac{c}{a}$

Equating the coefficients of x and the constant terms:

$$\boxed{\alpha+\beta=-\frac{b}{a} \qquad \text{and} \qquad \alpha\beta=\frac{c}{a}}$$

The quadratic equation can be written:

$$\boxed{\begin{array}{c} x^2-(\alpha+\beta)x+\alpha\beta=0 \\ \text{or} \\ x^2-(\text{sum of the roots})\,x+(\text{product of the roots})=0 \end{array}}$$

This can be used to obtain a new quadratic equation whose roots are known or are given as functions of α and β.

Example

a b c

If α and β are the roots of the equation $2x^2-6x+1=0$, find the value of:

(i) $\alpha+\beta$ **(ii)** $\alpha\beta$ **(iii)** $\alpha^2+\beta^2$ **(iv)** $\alpha^3\beta+\alpha\beta^3$

(v) $|\alpha-\beta|$ **(vi)** $\dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2}$ **(vii)** $\alpha^3+\beta^3$

Find a quadratic equation with roots $\dfrac{\alpha}{\beta}$ and $\dfrac{\beta}{\alpha}$ and write your answer in the form $px^2+qx+r=0$ where $p, q, r \in \mathbf{Z}$.

Solution:

$$2x^2-6x+1=0$$
$$x^2-3x+\tfrac{1}{2}=0 \qquad \text{(make coefficient of } x^2 \text{ equal to 1)}$$

(i) $\alpha+\beta=3$ **(ii)** $\alpha\beta=\frac{1}{2}$

What we do next is write each of the other expressions in terms of $(\alpha+\beta)$ and $\alpha\beta$ or use previous parts of the question.

(iii)
$$\begin{aligned}
\alpha+\beta&=3\\
(\alpha+\beta)^2&=(3)^2\\
\alpha^2+2\alpha\beta+\beta^2&=9\\
\alpha^2+\beta^2&=9-2\alpha\beta\\
\alpha^2+\beta^2&=9-2(\tfrac{1}{2})\\
\alpha^2+\beta^2&=9-1=8
\end{aligned}$$

(iv)
$$\begin{aligned}
&\alpha^3\beta+\alpha\beta^3\\
&=\alpha\beta(\alpha^2+\beta^2) \qquad \text{(factorise)}\\
&=\tfrac{1}{2}(8)\\
&=4
\end{aligned}$$

(v)
$$\begin{aligned}
(\alpha-\beta)^2&=\alpha^2-2\alpha\beta+\beta^2\\
(\alpha-\beta)^2&=(\alpha^2+\beta^2)-2\alpha\beta\\
(\alpha-\beta)^2&=8-2(\tfrac{1}{2})\\
(\alpha-\beta)^2&=8-1\\
(\alpha-\beta)^2&=7\\
(\alpha-\beta)&=\pm\sqrt{7}\\
\therefore\quad |\alpha-\beta|&=\sqrt{7}
\end{aligned}$$

(vi)
$$\begin{aligned}
&\frac{1}{\alpha^2}+\frac{1}{\beta^2}\\
&=\frac{\alpha^2+\beta^2}{\alpha^2\beta^2}\\
&=\frac{\alpha^2+\beta^2}{(\alpha\beta)^2}\\
&=\frac{8}{(\frac{1}{2})^2}=\frac{8}{\frac{1}{4}}=32
\end{aligned}$$

(vii) $\alpha^3 + \beta^3$

$= (\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2)$ (factorise, sum of two cubes)

$= (\alpha + \beta)[(\alpha^2 + \beta^2) - \alpha\beta]$ (group into previous expressions)

$= (3)[(8) - \tfrac{1}{2}]$

$= (3)(7\tfrac{1}{2}) = 22\tfrac{1}{2}$

$x^2 -$ (sum of the roots)$x +$ (product of the roots) $= 0$

$$x^2 - \left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right)x + \left(\frac{\alpha}{\beta}\right)\left(\frac{\beta}{\alpha}\right) = 0$$

$$x^2 - \left(\frac{\alpha^2 + \beta^2}{\alpha\beta}\right)x + 1 = 0$$

$$x^2 - \left(\frac{8}{\frac{1}{2}}\right)x + 1 = 0$$

$$x^2 - 16x + 1 = 0$$

$$\boxed{\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta}}$$

$$\boxed{\begin{aligned}\alpha^2 + \beta^2 &= 8 \\ \alpha\beta &= \tfrac{1}{2}\end{aligned}}$$

Example

One root of the equation $ax^2 + bx + c = 0$ is five times the other.
Show that $5b^2 = 36ac$, $a \neq 0$.

Solution:

Let the roots be α and 5α.

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \quad \text{(make coefficient of } x^2 \text{ equal to 1)}$$

$$\therefore \quad \alpha + 5\alpha = -\frac{b}{a} \quad \text{and} \quad (\alpha)(5\alpha) = \frac{c}{a}$$

$$6\alpha = -\frac{b}{a} \quad ① \quad \text{and} \quad 5\alpha^2 = \frac{c}{a} \quad ②$$

α is in both equations and not in the solution required.
Therefore, get α on its own from ① and put this into ②.

$$6\alpha = -\frac{b}{a} \quad ①$$

$$\alpha = \left(-\frac{b}{6a}\right)$$

put this into ②

$$5\alpha^2 = \frac{c}{a}$$

$$5\left(-\frac{b}{6a}\right)^2 = \frac{c}{a}$$

$$5\left(\frac{b^2}{36a^2}\right) = \frac{c}{a}$$

$$\frac{5b^2}{36a^2} = \frac{c}{a}$$

$$5ab^2 = 36a^2c$$

$$5b^2 = 36ac$$

Exercise 2.4

1. If α and β are the roots of the equation $x^2 + 2x + 5 = 0$, find the value of:
   (i) $\alpha + \beta$ **(ii)** $\alpha\beta$ **(iii)** $\alpha^2\beta + \alpha\beta^2$ **(iv)** $\alpha^2 + \beta^2$
   (v) $\alpha^3\beta + \alpha\beta^3$ **(vi)** $(\alpha - \beta)^2$ **(vii)** $\alpha^3 + \beta^3$ **(viii)** $(\alpha^2 - \beta)(\beta^2 - \alpha)$
   (ix) $\frac{1}{\alpha} + \frac{1}{\beta}$ **(x)** $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ **(xi)** $\frac{2\beta}{1 + \frac{\beta}{\alpha}}$ **(xii)** $\frac{1}{\alpha\beta} - \frac{1}{\beta} - \frac{1}{\alpha}$

2. If α and β are the roots of the equation $x^2 - 4x - 3 = 0$, find the value of:
   (i) $3\alpha + 3\beta$ **(ii)** $\alpha^2\beta^2$ **(iii)** $\alpha^2 + \beta^2$ **(iv)** $\alpha^3 + \beta^3$
   (v) $\frac{4}{\alpha} + \frac{4}{\beta}$ **(vi)** $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ **(vii)** $\alpha(1 + \beta) + \beta(1 + \alpha)$ **(viii)** $\left(\alpha + \frac{1}{\beta}\right)\left(\beta + \frac{1}{\alpha}\right)$

   Form a quadratic equation, with integer coefficients, whose roots are:
   (ix) $2\alpha, 2\beta$ **(x)** α^2, β^2 **(xi)** $\alpha + 3, \beta + 3$ **(xii)** $3\alpha + 1, 3\beta + 1$
   (xiii) $\alpha - 1, \beta - 1$ **(xiv)** $\frac{1}{\alpha}, \frac{1}{\beta}$ **(xv)** $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ **(xvi)** $\alpha(1 - \beta), \beta(1 - \alpha)$

3. The roots of the equation $2x^2 + 6x + 3 = 0$ are α and β. Find the value of:
   (i) $\alpha + \beta$ **(ii)** $\alpha\beta$ **(iii)** $3\alpha + 3\beta$ **(iv)** $\alpha^2 + \beta^2$
   The roots of the equation $2x^2 + px + q = 0$ are $2\alpha + \beta$ and $2\beta + \alpha$.
   Find the value of p and the value of q.

4. The roots of the quadratic equation $2x^2 - 6x + 5 = 0$ are $(\alpha - 2)$ and $(\beta - 2)$.
   (i) Find the value of **(a)** $\alpha + \beta$ **(b)** $\alpha\beta$
   (ii) Form a quadratic equation, with integer coefficients, with roots α and β.

5. If α and β are the roots of the equation $x^2 - px + q = 0$, show that:
   (i) $\alpha^2 + \beta^2 = p^2 - 2q$ **(ii)** $(\alpha - \beta)^2 = p^2 - 4q$.

6. Given that one root of the equation $2x^2 - 12x + k = 0$ is twice the other root, find the value of k.

7. One root of the equation $x^2 - px + q = 0$ is twice the other.
   Show that $2p^2 = 9q$.

8. The equation $x^2 - 2px + q = 0$ has roots α and $\alpha + 2$.
   Verify that $p^2 = q + 1$.

9. The equation $x^2 - 12x + k = 0$ has roots α and α^2.
   Find the two possible values of k.

10. The equation $x^2 - ax + 16 = 0$ has roots α and α^3.
    Find the two possible values of a.

11. One root of the equation $ax^2 + bx + c = 0$ is three times the other.
    Show that $3b^2 = 16ac$.

12. One root of the equation $px^2 + qx + r = 0$ is four times the other.
    Show that $4q^2 - 25pr = 0$.

13. For what values of k is one of the roots of $x^2 - 4(k+1)x + (k^2 - k + 7) = 0$ equal to three tir other?

14. **(i)** The quadratic equation $x^2 + (2k+2)x + (2k+5) = 0$ has roots α and β.
Express, in terms of k, **(a)** $\alpha + \beta$ **(b)** $\alpha\beta$ **(c)** $2\alpha^2\beta + 2\alpha\beta^2$.
(ii) The equation $x^2 - px + q = 0$ has roots $2\alpha + \alpha\beta$ and $2\beta + \beta\alpha$.
(a) Show that $p = 6$.
(b) Express q in terms of k.
(c) Find the values of k for which $q = 0$.

Factor Theorem

A polynomial in x is a collection of powers of x added together.
For example, $2x^2 - 3x + 5$, $5x^3 + 6x^2 - x + 4$ are polynomials.

Note: **1.** There cannot be negative or fractional powers in a polynomial.
2. A polynomial is often denoted as $f(x)$.

Factor Theorem

If an algebraic expression is divided by one of its factors, then the remainder is zero. The expre $(x - k)$ is a factor of a polynomial $f(x)$, if the remainder when we divide $f(x)$ by $(x - k)$ is zero.

Generalising this:

1. If $f(k) = 0$, then $(x - k)$ is a factor of $f(x)$.
2. If $(x - k)$ is a factor of $f(x)$, then $f(k) = 0$.

The factor theorem can be extended:

1. If $f\left(\frac{b}{a}\right) = 0$, then $(ax - b)$ is a factor of $f(x)$.

2. If $(ax - b)$ is a factor of $f(x)$, then $f\left(\frac{b}{a}\right) = 0$.

The factor theorem can be used to factorise polynomials or to find unknown coefficients in a polyno

Here are some examples:

Factor	Put factor = 0 and solve	Factor Theorem
$x + 4$	$x = -4$	$f(-4) = 0$
$x - 3$	$x = 3$	$f(3) = 0$
$2x + 1$	$x = -\frac{1}{2}$	$f(-\frac{1}{2}) = 0$

Note: If $(x + a)$ and $(x + b)$ are both factors of a polynomial $f(x)$, then so is their pro $(x + a)(x + b) = x^2 + abx + ab$, also a factor, and vice versa.

Example

If $(2x - 1)$ is a factor of the polynomial $f(x) = 2x^3 - 5x^2 - kx + 3$, find the value of k. Hence, find the other two factors.

Solution:

$f(x) = 2x^3 - 5x^2 - kx + 3$

If $(2x - 1)$ is a factor, then $f(\frac{1}{2}) = 0$.

$$\begin{aligned}
f(\tfrac{1}{2}) &= 0 \\
2(\tfrac{1}{2})^3 - 5(\tfrac{1}{2})^2 - k(\tfrac{1}{2}) + 3 &= 0 \qquad \text{(replace } x \text{ with } \tfrac{1}{2}) \\
2(\tfrac{1}{8}) - 5(\tfrac{1}{4}) - k(\tfrac{1}{2}) + 3 &= 0 \\
\tfrac{1}{4} - \tfrac{5}{4} - \tfrac{1}{2}k + 3 &= 0 \\
1 - 5 - 2k + 12 &= 0 \qquad \text{(multiply each part by 4)} \\
-2k + 8 &= 0 \\
-2k &= -8 \\
k &= 4
\end{aligned}$$

Now divide $2x^3 - 5x^2 - 4x + 3$ by $(2x - 1)$

$$\begin{array}{r|l}
 & x^2 - 2x - 3 \\
2x - 1 & 2x^3 - 5x^2 - 4x + 3 \\
 & \underline{2x^3 - x^2} \\
 & \quad -4x^2 - 4x \\
 & \quad \underline{-4x^2 + 2x} \\
 & \qquad -6x + 3 \\
 & \qquad \underline{-6x + 3} \\
 & \qquad\qquad 0
\end{array}$$

Now factorise $x^2 - 2x - 3$

$x^2 - 2x - 3$

$= (x + 1)(x - 3)$

Thus, the other two factors are:

$(x + 1)$ and $(x - 3)$

Example

Let $f(x) = 2x^3 + mx^2 + nx + 2$ where m and n are constants.
Given that $x - 1$ and $x + 2$ are factors of $f(x)$, find the value of m and the value of n.

Solution:

$$f(x) = 2x^3 + mx^2 + nx + 2.$$

If $(x - 1)$ is a factor, then $f(1) = 0$.

$$\begin{aligned}
f(1) &= 0 \\
2(1)^3 + m(1)^2 + n(1) + 2 &= 0 \\
2 + m + n + 2 &= 0 \\
m + n &= -4 \quad ①
\end{aligned}$$

If $(x + 2)$ is a factor, then $f(-2) = 0$.

$$\begin{aligned}
f(-2) &= 0 \\
2(-2)^3 + m(-2)^2 + n(-2) + 2 &= 0 \\
-16 + 4m - 2n + 2 &= 0 \\
4m - 2n &= 14 \\
2m - n &= 7 \quad ②
\end{aligned}$$

We now solve the simultaneous equations ① and ②:

$$\begin{aligned}
m + n &= -4 \quad ① \\
\underline{2m - n} &\underline{= 7 \quad ②} \\
3m &= 3 \text{ (add)} \\
m &= 1
\end{aligned}$$

$$\begin{aligned}
m + n &= -4 \quad ① \\
1 + n &= -4 \\
n &= -5
\end{aligned}$$

Thus, $m = 1$ and $n = -5$

Example

$x^2 - px + q$ is a factor of $x^3 + 3px^2 + 3qx + r$.

(i) Show that $q = -2p^2$.
(ii) Show that $r = -8p^3$.

Solution:

Method 1: Equating the coefficients

Let $(x + k)$ be the third factor.

Thus,
$$(x+k)(x^2 - px + q) = x^3 + 3px^2 + 3qx + r$$
$$x^3 - px^2 + qx + kx^2 - kpx + qk = x^3 + 3px^2 + 3qx + r$$
$$x^3 + (-p+k)x^2 + (q-kp)x + qk = x^3 + 3px^2 + 3qx + r$$

Equating coefficients of like terms:

$-p + k = 3p$ ① $\quad q - kp = 3q$ ② $\quad qk = r$ ③

(Basic idea is to remove the constant k, which is not in the solution required)

$-p + k = 3p$ ① (get k on its own from ①)
$k = 4p$ (k on its own)

Put $k = 4p$ into ② and ③.

(i)
$$q - kp = 3q \quad ②$$
$$q - (4p)p = 3q$$
$$q - 4p^2 = 3q$$
$$-4p^2 = 2q$$
$$-2p^2 = q$$

(ii)
$$qk = r \quad ③$$
$$q(4p) = r$$
$$4pq = r$$
$$4p(-2p^2) = r \qquad \begin{pmatrix} q = -2p^2 \\ \text{from (i)} \end{pmatrix}$$
$$-8p^3 = r$$

Method 2: Using long division

$$\begin{array}{r|l} & x + 4p \\ \hline x^2 - px + q & x^3 + 3px^2 + 3qx + r \\ & x^3 - px^2 + qx \\ \hline & \quad 4px^2 + 2qx + r \\ & \quad 4px^2 - 4p^2x + 4pq \\ \hline & \quad (2q + 4p^2)x + (r - 4pq) \end{array}$$

Since $(x^2 - px + q)$ is a factor, the remainder must equal 0.

Thus
$$2q + 4p^2 = 0 \quad ①$$
$$2q = -4p^2$$
$$q = -2p^2$$
Put this into ②:

or
$$r - 4pq = 0 \quad ②$$
$$r - 4p(-2p^2) = 0$$
$$r + 8p^3 = 0$$
$$r = -8p^3$$

Exercise 2.5

1. Verify that $(x - 1)$ is a factor of $x^3 + 2x^2 - x - 2$ and find the other two factors.
2. Verify that $(x + 3)$ is a factor of $x^3 + 9x^2 + 23x + 15$ and find the other two factors.
3. Verify that $(2x - 1)$ is a factor of $6x^3 + 7x^2 - 9x + 2$ and find the other two factors.
4. Verify that $(2x - 3)$ is a factor of $2x^3 - 15x^2 + 34x - 24$ and find the other two factors.
5. Verify that $(x - 1)$ is a factor of $x^3 - (2k + 1)x^2 + (k^2 + 2k)x - k^2$.
6. If $(x + 2)$ is a factor of the polynomial $f(x) = 6x^3 + kx^2 + 11x - 6$, find the value of k. Hence, find the other two factors.
7. If $(2x + 1)$ is a factor of the polynomial $f(x) = 2x^3 + 7x^2 + kx + 2$, find the value of k. Hence, find the other two factors.
8. Let $f(x) = px^3 + 3x^2 - 9x + q$ where p and q are constants. Given that $(x + 1)$ and $(x - 2)$ are factors of $f(x)$, find the value of p and the value of q.
9. Let $p(x) = 2x^3 - ax^2 - bx + 42$ where a and b are constants. Given that $(x - 2)$ and $(x + 3)$ are factors of $p(x)$, find the value of a and the value of b.
10. Let $f(x) = 2x^3 + ax^2 + bx - 6$ where a and b are constants. Given that $f(-2) = 0$ and $f(\frac{1}{2}) = 0$, find the value of a and the value of b.
11. Let $f(x) = x^3 - (h + 2)x + 2k$ and $p(x) = 2x^3 + hx^2 - 4x - k$. Given that $(x + 3)$ is a common factor of $f(x)$ and $p(x)$, find the value of h and the value of k.
12. Factorise $x^2 + x - 6$.
    Let $f(x) = px^3 + x^2 - 20x + q$ where p and q are constants.
    Given that $x^2 + x - 6$ is a factor of $f(x)$, find the value of p and the value of q.
13. Given that $px^3 + 8x^2 + qx + 6$ is exactly divisible by $x^2 - 2x - 3$, find the value of p and the value of q.
14. If $(x - 2)^2$ is a factor of $x^3 + px + q$, find the value of p and the value of q.
15. $(x - a)^2$ is a factor of $x^3 + 3px + q$.
    Show that: **(i)** $p = -a^2$ **(ii)** $q = 2a^3$.
16. $x^2 + bx + c$ is a factor of $x^3 - p$. Show that:
    (i) $c = b^2$ **(ii)** $bc = p$ **(iii)** $b^3 = p$ **(iv)** $c^3 = p^2$.
17. $x^2 - px + 1$ is a factor of $ax^3 + bx + c$ where $a \neq 0$. Show that:
    (i) $p = \dfrac{c}{a}$ **(ii)** $c^2 = a(a - b)$.

Solving Cubic Equations

Any equation of the form $ax^3 + bx^2 + cx + d = 0$, $a \neq 0$, is called a **cubic** equation.
We use the factor theorem to find one root, and hence one factor.

A cubic equation is solved with the following steps:

1. Find the first root k by trial and error, i.e. try $f(1), f(-1), f(2), f(-2)$, etc. (Only try numbers that divide evenly into the constant in the equation.)
2. If $x = k$ is a root, then $(x - k)$ is a factor.
3. Divide $f(x)$ by $(x - k)$ which always gives a quadratic expression.
4. Let this quadratic = 0 and solve by factors or formula.

Note: Each cubic equation we are asked to solve must have at least one integer root.

Example

Solve the equation $2x^3 + x^2 - 13x + 6 = 0$.

Solution:

Let $f(x) = 2x^3 + x^2 - 13x + 6$.

1. The first root will be a factor of 6.

$\therefore$ We need try only those values which are factors of 6, i.e. $\pm 1, \pm 2, \pm 3, \pm 6$.

$$f(1) = 2(1)^3 + (1)^2 - 13(1) + 6 = 2 + 1 - 13 + 6 = -4 \neq 0$$

$$f(-1) = 2(-1)^3 + (-1)^2 - 13(-1) + 6 = -2 + 1 + 13 + 6 = 18 \neq 0$$

$$f(2) = 2(2)^3 + (2)^2 - 13(2) + 6 = 16 + 4 - 26 + 6 = 0$$

$\therefore$ $x = 2$ is a root

2. $\therefore$ $x - 2$ is a factor

3. Divide $(2x^3 + x^2 - 13x + 6)$ by $(x - 2)$

$$\begin{array}{r|l} & 2x^2 + 5x - 3 \\ \hline x - 2 & 2x^3 + x^2 - 13x + 6 \\ & \underline{2x^3 - 4x^2} \\ & \quad 5x^2 - 13x \\ & \quad \underline{5x^2 - 10x} \\ & \qquad -3x + 6 \\ & \qquad \underline{-3x + 6} \\ & \qquad\qquad 0 \end{array}$$

4. Let $2x^2 + 5x - 3 = 0$

$(2x - 1)(x + 3) = 0$

$2x - 1 = 0$ or $x + 3 = 0$

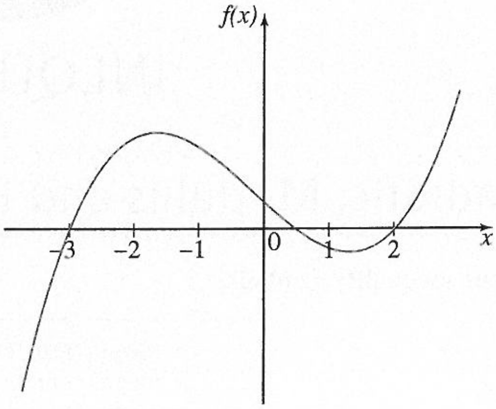
$x = \frac{1}{2}$ or $x = -3$

Thus, the three roots of the equation

$2x^3 + x^2 - 13x + 6 = 0$

are -3, $\frac{1}{2}$ and 2.

Rough graph of $f(x) = 2x^3 + x^2 - 13x + 6$:



Note: If we draw the graph of $f(x) = 2x^3 + x^2 - 13x + 6$ we can see that the roots of the equation $f(x) = 0$ occur where the graph of $f(x)$ cuts the x-axis.

Exercise 2.6

1. Find the three linear factors of $x^3 - x^2 - 14x + 24$.
   Hence, solve the equation $x^3 - x^2 - 14x + 24 = 0$.

2. Factorise $2x^3 - x^2 - 2x + 1$. Hence solve the equation $2x^3 + 1 = x^2 + 2x$.

3. Show that $x = \frac{1}{2}$ is a root of the equation $2x^3 - 5x^2 - 4x + 3 = 0$ and find the other two roots.

4. Show that $x = 2$ is a root of the equation $x^3 + 4x^2 - 11x - 2 = 0$ and find the other two roots, giving your answer in the form $a \pm b\sqrt{b}$.

5. If $x = -\frac{1}{2}$ is one root of the equation $2x^3 - 9x^2 + kx + 6 = 0$, find the value of k.
   Find the other two roots of the equation.

6. Let $p(x) = ax^3 - 5x^2 - bx + 18$.
   If -2 and 3 are roots of the equation $p(x) = 0$, find the value of a and the value of b.
   If $p(k) = 0$, $k \neq -2, 3$, find the value of k.

7. If k is a root of the equation $3x^3 + (k + 3)x^2 + (7 - k - 4k^2)x - 4 = 0$, find the values of k.

8. Find the values of the constants p, q and r for which $(x - 4)(x - 2)(x + p) = x^3 - 7x^2 + qx + r$ for all values of $x \in \mathbf{R}$. Using these values of q and r solve the equation $x^3 - 7x^2 + qx + r = 0$.

9. Verify that $-4p$ is a root of the equation $x^3 + 3px^2 - 6p^2x - 8p^3 = 0$. Hence, or otherwise, find the three roots of $x^3 + 3px^2 - 6p^2x - 8p^3 = 0$ in terms of p.