CHAPTER 5

PERMUTATIONS, COMBINATIONS AND PROBABILITY

Operations

The result of an operation is called an '**outcome**'.
For example, if we throw a die one possible outcome is 5.
If we throw a die there are 6 possible outcomes: 1, 2, 3, 4, 5 or 6.

Fundamental Principle of Counting 1

Suppose one operation has m possible outcomes and that a second operation has n outcomes. The number of possible outcomes when performing the first operation **followed by** the second operation is $m \times n$.

Performing one operation **and** another operation means we **multiply** the number of possible outcomes.

(first operation)	**AND**	(second operation)
↓	↓	↓
m	$\times$	n

'**And**' is understood to mean '**multiply**'.

Note: We assume that the outcome of one operation does not affect the number of possible outcomes of the other operation.

The fundamental principle 1 of counting can be extended to three or more operations.

Example

(i) If a die is thrown and a coin is tossed, how many different outcomes are possible?
(ii) Write out all the possible outcomes.

Solution:

Die and Coin

(i) Represent each operation with an empty box: $\square \times \square$

1. There are 6 possible outcomes for a die: 1, 2, 3, 4, 5 or 6.
2. There are 2 possible outcomes for a coin: H or T.

Hence, the number of different outcomes $= \boxed{6} \times \boxed{2} = 12$.

(ii)

T	•	•	•	•	•	•
H	•	•	•	•	•	•
	1	**2**	**3**	**4**	**5**	**6**

$(1, H), (2, H), (3, H), (4, H), (5, H), (6, H)$
$(1, T), (2, T), (3, T), (4, T), (5, T), (6, T)$

Note: It can help to write down one possible outcome above the box.

	Die	and	Coin		
One possible outcome:	5	↓	T		
Number of outcomes:	$\boxed{6}$	×	$\boxed{2}$	=	12

This is very useful when trying to decide the number of possible outcomes at a particular stage, especially when certain choices are restricted. For example, the letter v cannot be in the second place, or the number must be even.

Fundamental Principle of Counting 2

Suppose one operation has m possible outcomes and that a second operation has n outcomes. Then the number of possible outcomes of the first operation **or** the second operation is given by $m + n$.

Performing one operation **or** another operation means we **add** the number of possible outcomes.

(first operation)	**OR**	(second operation)
↓	↓	↓
m	+	n

'Or' is understood to mean **'add'**.

Note: We assume it is not possible for both operations to occur. In other words, there is no overlap of the two operations.

The fundamental principle 2 can be extended to three or more operations, as long as none of the operations overlap.

Example

A bag contains nine discs, numbered from 1 to 9. A disc is drawn from the bag.
If the number is even, then a coin is tossed. If the number is odd, then a die is thrown.
How many outcomes are possible?

Solution:

Break the experiment into two different experiments and work out the number of outcomes separately. Then add these results.

$$\begin{array}{ccccc} & \textbf{First experiment} & & \textbf{Second experiment} \\ & \left[\begin{pmatrix}\text{even}\\ \text{number}\end{pmatrix} \text{ and } \begin{pmatrix}\text{toss a}\\ \text{coin}\end{pmatrix}\right] & \text{or} & \left[\begin{pmatrix}\text{odd}\\ \text{number}\end{pmatrix} \text{ and } \begin{pmatrix}\text{throw}\\ \text{a die}\end{pmatrix}\right] \\ & \downarrow & \downarrow & \downarrow \\ = & \boxed{4} \times \boxed{2} & + & \boxed{5} \times \boxed{6} \\ = & 8+30 & & \\ = & 38 & & \end{array}$$

Permutations (Arrangements)

> A permutation is an arrangement of a number of objects in a definite order.

Consider the three letters P, Q and R. If these letters are written down in a row, there are six different possible arrangements:

$$PQR \text{ or } PRQ \text{ or } QPR \text{ or } QRP \text{ or } RPQ \text{ or } RQP$$

There is a choice of 3 letters for the first place, then there is a choice of 2 letters for the second place and there is only 1 choice for the third place.
Thus the three operations can be performed in $\boxed{3} \times \boxed{2} \times \boxed{1} = 6$ ways.
The boxes are an aid in helping to fill in the number of ways each choice can be made at each position.
In an arrangement, or permutation, the order of the objects chosen is important.

> If we have n **different** objects to arrange, then:
> The total number of arrangements $= n!$
> $n! = n(n-1)(n-2)(n-3) \cdots \times 3 \times 2 \times 1$

For example, $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$
Using a calculator: 6 $\boxed{n!}$ $\boxed{=}$ 720
Suppose we have 5 different objects, and we want to find the number of possible arrangements taking 3 objects at a time. We could use the fundamental principle of counting 1, i.e.

$$\begin{array}{ccccccc} \text{1st} & \text{and} & \text{2nd} & \text{and} & \text{3rd} & & \\ \boxed{5} & \times & \boxed{4} & \times & \boxed{3} & = & 60 \end{array}$$

However, we can also do this type of calculation using factorials.

$$5 \times 4 \times 3 = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = \frac{5!}{2!} = \frac{5!}{(5-3)!}$$

Similarly:

$$6 \times 5 = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1} = \frac{6!}{4!} = \frac{6!}{(6-2)!}$$

$$7 \times 6 \times 5 \times 4 = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = \frac{7!}{3!} = \frac{7!}{(7-4)!}$$

Notice that in each case the number of arrangements is given by:

$$\frac{(\text{Total number of objects})!}{(\text{Total number of objects} - \text{number of objects to be arranged})!}$$

> The number of arrangements of n different objects taking r at a time is:
>
> $$^{n}p_{r} = \frac{n!}{(n-r)!} = n(n-1)(n-2) \cdot \cdots \cdot (n-r+1)$$

Note: There is an '*nPr*' button on most calculators.

Example

How many arrangements can be made of the letters P, Q, R, S, T, taking two letters at a time, if no letter can be repeated?

Solution:

Method 1: (using the fundamental principle 1 of counting)

1st	and	2nd		
$\boxed{5}$	$\times$	$\boxed{4}$	=	20

Method 2: (using factorials)

We have 5 letters and want to arrange two.

$$^{5}P_{2} = \frac{5!}{(5-2)!} = \frac{5!}{3!} = \frac{120}{6} = 20$$

Example

How many different arrangements can be made from the letters V, W, X, Y, Z, taking all the letters at a time, if V must be second and Z can never be last?

Solution:

Represent each choice with a box:

	1st		2nd		3rd		4th		5th		
Possible outcome:	W		V		Z		Y		X		
Number of ways:	$\boxed{3}$	$\times$	$\boxed{1}$	$\times$	$\boxed{2}$	$\times$	$\boxed{1}$	$\times$	$\boxed{3}$	=	18

(most restrictive, must be V) (second most restrictive, cannot use V or Z)

Start with the choice that has most restrictions and choose a possible letter each time.

In some questions we make use of the following fact:

$$\left(\begin{array}{c}\text{The number of arrangements in which}\\ \text{an outcome does \textbf{not} occur}\end{array}\right) = \left(\begin{array}{c}\text{Total number}\\ \text{of arrangements}\end{array}\right) - \left(\begin{array}{c}\text{The number of arrangements}\\ \text{in which the outcome \textbf{does} occur}\end{array}\right)$$

Example

A, B, C, D, E and F are six students. In how many ways can they be seated in a row if:

(i) there are no restrictions on the seating;
(ii) A and B must sit beside each other;
(iii) A and B must not sit beside each other;
(iv) D, E and F must sit beside each other;
(v) A and F must sit at the end of each row?

Solution:

(i) no restrictions

number of arrangements $= \boxed{6} \times \boxed{5} \times \boxed{4} \times \boxed{3} \times \boxed{2} \times \boxed{1} = 6! = 720$.

(ii) A and B must sit beside each other

Consider A and B as one person.

$\boxed{A, B}, \boxed{C}, \boxed{D}, \boxed{E}, \boxed{F}$

The 5 persons can be arranged in 5! ways.
But A and B can be arranged in 2! ways while seated together (i.e. AB or BA).
Thus, the number of arrangements $= 2! \times 5! = 2 \times 120 = 240$.

(iii) A and B must not sit beside each other

$$\left(\begin{array}{c}\text{Number of arrangements with}\\ A \text{ and } B \text{ not together}\end{array}\right) = \left(\begin{array}{c}\text{Total number}\\ \text{of arrangements}\end{array}\right) - \left(\begin{array}{c}\text{Number of arrangements}\\ \text{with } A \text{ and } B \text{ together}\end{array}\right)$$

$$= 720 - 240 = 480$$

(iv) D, E and F must sit beside each other

Consider D, E and F as one person.

$\boxed{A}, \boxed{B}, \boxed{C}, \boxed{D, E, F}$

The 4 persons can be arranged in 4! ways.
But D, E and F can be arranged in 3! ways while seated together.
Thus, the number of arrangements $= 3! \times 4! = 6 \times 24 = 144$.

(v) A and F must sit at the end of each row

Put A and F at the ends and B, C, D and E in between them.

$\boxed{A}, \boxed{B, C, D, E}, \boxed{F}$

B, C, D and E can be arranged in 4! ways while seated together.
A and F can exchange places.
Thus, the first and last can be arranged in 2! ways.
Thus, the number of arrangements $= 2! \times 4! = 2 \times 24 = 48$.

Example

How many different four-digit numbers greater than 6,000 can be formed using the digits 1, 2, 4, 5, 6, 7 if **(i)** no digit can be repeated; **(ii)** repetitions are allowed?

Solution:

Represent each choice with a box. The number must be greater than 6,000; thus, the first place can be filled in only 2 ways (with 6 or 7). Fill this in first, then fill the other places. Only use 4 boxes, as our choice is restricted to 4 digits at a time.

(i) No digit can be repeated

$\boxed{2} \times \boxed{5} \times \boxed{4} \times \boxed{3} = 120$

(ii) Repetitions are allowed

$\boxed{2} \times \boxed{6} \times \boxed{6} \times \boxed{6} = 432$

Example

How many different five-digit numbers can be formed from the digits, 1, 2, 3, 4 and 5 if:

(i) there are no restrictions on digits and repetitions are allowed;

(ii) the number is odd and no repetitions are allowed;

(iii) the number is even and repetitions are allowed;

(iv) the number is greater than 50,000 and no repetitions are allowed?

Solution:

Represent each choice with a box.

(i) no restrictions and repetitions allowed

$\boxed{5} \times \boxed{5} \times \boxed{5} \times \boxed{5} \times \boxed{5} = 3{,}125$

(ii) must be odd and no repetitions

Thus, the last place can be filled in only 3 ways (1, 3 or 5). Fill this in first, then fill in the other places.

$\boxed{4} \times \boxed{3} \times \boxed{2} \times \boxed{1} \times \boxed{3} = 72$

(iii) must be even and repetitions allowed

Thus, the last place can be filled in only 2 ways (2 or 4). Fill this in first, then fill in the other places.

$\boxed{5} \times \boxed{5} \times \boxed{5} \times \boxed{5} \times \boxed{2} = 1{,}250$

(iv) must be greater than 50,000 and no repetitions

Thus, the first place can be filled in only 1 way (with 5). Fill this in first, then fill in the other places.

$\boxed{1} \times \boxed{4} \times \boxed{3} \times \boxed{2} \times \boxed{1} = 24$

Exercise 5.1

1. A game consists of spinning an unbiased, five-sided spinner which can land on A, B, C, D or E, and throwing an unbiased die. How many different outcomes of the game are possible?

2. A fifth-year student must choose one subject from each of the following three groups:
   Group 1: Music, Technical Graphics, Applied Mathematics or Classical Studies
   Group 2: Physics, Chemistry or Biology
   Group 3: Economics or Accounting
   In how many ways can the student choose the three subjects?

3. A bag contains five discs, numbered from 1 to 5 inclusive. A disc is drawn from the bag. If the number is even, then a die is thrown. If the number is odd, then a coin is tossed. How many outcomes are possible?

4. A bag contains seven discs, numbered from 1 to 7 inclusive. A disc is drawn at random from the bag and **not** replaced. If the number is even, a second disc is drawn from the bag. If the first number is odd, then a die is thrown. How many outcomes are possible?

5. How many different arrangements can be made from the letters P, Q, R, S, T, if no letter can be repeated and taking:
   (i) five letters **(ii)** four letters **(iii)** three letters **(iv)** two letters, at a time?

6. Ten horses run in a race. In how many ways can the first, second and third places be filled if there are no dead heats?

7. How many different arrangements can be made using all the letters of the word *DUBLIN*?
   (i) How many arrangements begin with the letter *D*?
   (ii) How many arrangements begin with B and end in *L*?
   (iii) How many arrangements begin with a vowel?
   (iv) How many arrangements begin and end with a vowel?
   (v) How many arrangements end with *LIN*?
   (vi) How many arrangements begin with *D* and end in *LIN*?

8. Taking all the letters of the word *ALGEBRA*, in how many arrangements are the two *A*s together?

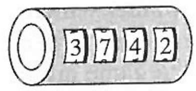
9. **(i)** In how many ways can three girls and two boys be seated in a row of 5 seats?
   (ii) In how many ways can this be done if the boys must sit together?
   (iii) In how many ways can this be done if the boys must not sit together?

10. How many arrangements of the letters of the word *FORMULAS* are possible if:
    (i) all letters are used in the arrangement?
    (ii) the three vowels must come together in the arrangement?
    (iii) the three vowels must not **all** come together in the arrangement?

11. Six children are to be seated in a row on a bench.
    (i) How many arrangements are possible?
    (ii) How many arrangements are possible if the youngest child must sit at the left-hand end and the oldest child must sit at the right-hand end?
    If two of the children are twins, in how many ways can the children be arranged if:
    (iii) the twins are together **(iv)** the twins are not together?

12. In how many ways can four letters of the word *NUMBER* be arranged in a row, if no letter is repeated?

13. A permutation lock has four rings which can be rotated about an axle. There are 10 digits (0, 1, 2, . . . , 8, 9) on each ring. If no digit can be repeated and 0 can never be first, find the maximum number of such locks that could be manufactured if no two locks have the same code, and the lock will open only when a certain code of 4 digits is in line (see diagram).



14. Raffle tickets are printed in three colours, Purple, Green or Yellow. Each ticket contains a letter followed by two digits (e.g. Purple Q47, Yellow K00). How many different tickets can be printed?

15. A number-plate is to consist of three letters of the English alphabet and two digits. If no letter or digit can be repeated and 0 can never be used as the first digit, how many different plates can be manufactured?

(an example)

16. How many different three-digit numbers can be formed using the digits 0, 1, 2, 3, 4 if 0 cannot be the first digit and:
(i) no digit may be repeated **(ii)** repetitions are allowed?

17. How many four-digit numbers can be formed from the digits 1, 2, 3, 5, 6 and 8 if:
(i) there are no restrictions and repetitions are allowed;
(ii) the number is odd and no repetitions are allowed;
(iii) the number is divisible by 5 and repetitions are allowed;
(iv) the number is greater than 5,000, divisible by 5 and no repetitions are allowed?

18. How many numbers between 2,000 and 4,000 can be made with the digits 1, 2, 3 and 4, if no digit may be repeated?

19. How many numbers between 100 and 1,000 use only odd digits, if no digits are repeated?

20. How many odd numbers between 2,000 and 3,000 can be formed from the digits 1, 2, 3, 4, 5 and 6, if:
(i) repetitions are allowed **(ii)** repetitions are not allowed?

21. **(i)** How many different numbers, each with 3 digits or fewer, can be formed from the digits 2, 3, 4, 5, 6? Each digit can be used only once in each number.
(ii) How many of the above numbers are odd?

22. How many odd numbers between 4,000 and 6,000 can be formed from the digits 3, 4, 5, 6, 7 and 8, if no digit can be repeated?

23. How many different three-digit numbers can be formed using the digits 0 to 9 inclusive, if no digit can be used more than once and 0 cannot be the first digit?
(i) If one of the digits must be 7, how many different numbers can be formed?
(ii) If the digits 2 and 5 cannot be used together, how many different numbers can be formed?

Combinations (Selections)

A combination is a selection of a number of objects in any order.

In making a selection of a number of objects from a given set, only the contents of the group selected are important, not the order in which the items are selected.

For example, AB and BA represent the same selection.
However, AB and BA represent different arrangements.

Note: What is called a 'combination lock' should really be called a 'permutation lock', as the order of the digits is essential.

The $\binom{n}{r}$ Notation

$\binom{n}{r}$ gives the number of ways of choosing r objects from n different objects.

Its value can be calculated in two ways:

1. $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$ (definition)

2. $\binom{n}{r} = \dfrac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$ (in practice)

Both give the same result; however, the second is easier to use in practical questions.
For example:

1. $\binom{6}{2} = \dfrac{6!}{2!(6-2)!} = \dfrac{6!}{2!4!} = \dfrac{720}{2 \times 24} = 15$

2. $\binom{6}{2} = \dfrac{6.5}{2.1}$ $\begin{array}{l} \rightarrow \text{ start at 6, go down two terms} \\ \hline \rightarrow \text{ start at 2, go down two terms} \end{array}$

$= 15$

Notes: **1.** $\binom{n}{r}$ is pronounced 'n-c-r' or 'n-choose-r'.

2. $\binom{n}{0} = 1$, i.e., there is only one way of choosing no objects out of n objects.

3. $\binom{n}{n} = 1$, i.e., there is only one way of choosing n objects out of n objects.

4. $\binom{n}{r} = \binom{n}{n-r}$; use this when r is greater than $\dfrac{n}{2}$.

Explanation for Note 4:

Let's assume you have 13 soccer players and you can pick only 11 to play.

The number of ways of choosing 11 from 13 is given by $\binom{13}{11}$.

$$\binom{13}{11} = \frac{13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3}{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = 78$$

However, every time you choose 11 to play, you choose 2 who cannot play.

Thus $\binom{13}{11} = \binom{13}{2} = \frac{13 \times 12}{2 \times 1} = 78$ (same as before).

Notice that $11 + 2 = 13$

Similarly, $\binom{20}{17} = \binom{20}{3}$ as $17 + 3 = 20$

and $\binom{100}{98} = \binom{100}{2}$ as $98 + 2 = 100$

If r is large, your calculator may not be able to do the calculation: thus use $\binom{n}{r} = \binom{n}{n-r}$.

Note: $\binom{n}{r}$ is sometimes written as ${}^nC_r$ or ${}_nC_r$.

Example

Ten people take part in a chess competition. How many games will be played if every person must play each of the others?

Solution:

We **have** 10 people to choose from, of whom we want to **choose** 2 (as 2 people play in each game). Thus, $n = 10$, $r = 2$.

$$\text{Number of games} = \binom{10}{2} = \frac{10 \times 9}{2 \times 1} = 45.$$

Example

(i) In how many ways can a committee of 4 people be chosen from a panel of 10 people?
(ii) If a certain person must be on the committee, in how many ways can the committee be chosen?
(iii) If a certain person must not be on the committee, in how many ways can the committee be chosen?

Solution:

(i) We **have** a panel of 10 people to choose from, and we need to **choose** a committee of 4.

$$\therefore \quad n = 10, \; r = 4$$

$$\binom{10}{4} = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210$$

Thus, from a panel of 10 people, we can choose 210 different committees of 4 people.

(ii) One particular person **must** be on the committee.
Thus, we **have** a panel of 9 people to choose from, and we need to **choose** 3 (as one person is already chosen).

$$\therefore \quad n=9,\ r=3$$

$$\binom{9}{3}=\frac{9\times 8\times 7}{3\times 2\times 1}=84$$

Thus, from a panel of 10 people, we can choose 84 different committees of 4 people, if one particular person of the 10 must be on every committee.

(iii) One particular person **must not** be on the committee.
Thus, we **have** a panel of 9 to choose from (as one person cannot be chosen), and we need to **choose** 4.

$$\therefore \quad n=9,\ r=4$$

$$\binom{9}{4}=\frac{9\times 8\times 7\times 6}{4\times 3\times 2\times 1}=126$$

Thus from a panel of 10 people, we can choose 126 different committees of 4 people, if one particular person of the 10 must not be on the committee.

Example

(i) In how many ways can a group of five be selected from nine people?
(ii) How many groups can be selected if two particular people from the nine cannot be in the same group?

Solution:

(i) We have nine from whom we want to choose five. Thus, $n=9$, $r=5$.

$$\binom{9}{5}=\frac{9\times 8\times 7\times 6\times 5}{5\times 4\times 3\times 2\times 1}=126$$

(ii) In order to calculate how many groups of 5 can be selected if two particular people cannot be included, we first need to calculate the number of ways of selecting 5 people with these particular two people always included, i.e. we have 7 from whom we choose 3 (because two are already selected). Thus $n=7$, $r=3$.

$$\binom{7}{5}=\frac{7\times 6\times 5}{3\times 2\times 1}=35$$

$$\left(\begin{array}{c}\text{The number of ways of selecting}\\ \text{a group of 5 people from 9}\\ \text{when two particular people are}\\ \text{not to be in the same group}\end{array}\right)=\left(\begin{array}{c}\text{Total number}\\ \text{of ways of}\\ \text{selecting a group}\\ \text{of 5 from 9}\end{array}\right)-\left(\begin{array}{c}\text{The number of ways of}\\ \text{selecting a group of 5}\\ \text{people from 9 with}\\ \text{these two particular people}\end{array}\right)$$

$$=126-35$$

$$=91$$

Example

In how many ways may 10 people be divided into three groups of 5, 3 and 2 people?

Solution:

The first group of five can be selected in $\binom{10}{5} = 252$ ways.

Once this is done, the second group of 3 can be selected in $\binom{5}{3} = 10$ ways.

The remaining 2 people can now be selected in $\binom{2}{2} = 1$ way.

Thus, the total number of ways 10 people can be divided into groups of 5, 3 and 2:

$= \binom{10}{5} \times \binom{5}{3} \times \binom{2}{2} = 252 \times 10 \times 1 = 2{,}520$

Example

Four letters are selected from the word *SECTIONAL*.

(i) How many different selections are possible?

(ii) How many of these selections contain at least one vowel?

Solution:

(i) We have 9 different letters to choose from and we need to select 4,

$$\therefore \quad n = 9, r = 4$$

$$\binom{9}{4} = \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} = 126$$

(ii) Split the word *SECTIONAL* up into vowels and consonants.

The four vowels are E, I, O, A and the five consonants are S, C, T, N, L.

'At least one vowel' means one vowel, two vowels, three vowels or four vowels.

It is easier to calculate the number of selections containing no vowels (i.e., four consonants), and subtract this from the number of ways of selecting four letters without any restrictions.

Number of selections containing no vowels

= Number of selections containing four consonants $= \binom{5}{4} = \frac{5 \times 4 \times 3 \times 2}{4 \times 3 \times 2 \times 1} = 5.$

(There are five consonants, and we want to select four, thus $n = 5$ and $r = 4$.)

In every other selection there must be at least one vowel.

Thus, the number of selections containing at least one vowel

= (total number of selections) – (number of selections containing no vowels)

$= 126 - 5$

$= 121$

Sometimes we have to deal with problems where objects are chosen from two different groups. This involves choosing a number of objects from one group **AND** then choosing a number of objects from the other group.

Notes: There are two key words when applying the fundamental principle of counting:

1. '**And**' is understood to mean '**multiply**'. Thus, and $= \times$.
2. '**Or**' is understood to mean '**add**'. Thus, or $= +$.

Example

How many different basketball teams, each consisting of 3 boys and 2 girls, can be formed from 7 boys and 5 girls?

Solution:

And $= \times$	Or $= +$

We have 7 boys and 5 girls. These are the upper numbers in the combination bracket.
A team must consist of 5 players.
We need to choose '3 boys and 2 girls'. These are the lower numbers in the combination bracket.

Number of ways of choosing 3 boys from 7 boys $= \binom{7}{3} = 35$

Number of ways of choosing 2 girls from 5 girls $= \binom{5}{2} = 10$

$\therefore$ the number of basketball teams consisting of 3 boys and 2 girls $= 35 \times 10 = 350$.

Example

There are 5 women and 4 men in a club. A team of four has to be chosen. How many different teams can be chosen if there must be either exactly one woman or exactly two women on the team?

Solution:

And $= \times$	Or $= +$

We have 5 women and 4 men and these are **always** the upper numbers in the combination bracket.
A team must consist of 4 people.
Thus, exactly one woman on the team means '1 woman **and** 3 men';
and exactly two women on the team means '2 women **and** 2 men'.
Thus, we need to choose '1 woman **and** 3 men' **or** '2 women **and** 2 men'.
Let W stand for women and let M stand for men.

$$
\begin{array}{lccccccccc}
 & \text{1W} & \text{and} & \text{3M} & \text{or} & \text{2W} & \text{and} & \text{2M} & & \text{(lower numbers in each case)} \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\
 & \binom{5}{1} & \times & \binom{4}{3} & + & \binom{5}{2} & \times & \binom{4}{2} & & \\
= & 5 & \times & 4 & + & 10 & \times & 6 & & \\
= & 20+60 & & & & & & & & \\
= & 80 & & & & & & & &
\end{array}
$$

Thus, 80 teams can have either one woman or two women on the team.

Example

How many bundles of 5 different books can be made from 8 Maths books and 6 Physics books, if the number of Maths books must always be greater than the number of Physics books?

Solution:

We have 8 Maths books and 6 Physics books and these are **always** the upper numbers in the combination bracket.
A bundle must consist of 5 books. We need to have more Maths books than Physics books.
Therefore, we need to choose:
(5 Maths and 0 Physics books) **or** (4 Maths and 1 Physics books) **or** (3 Maths and 2 Physics books).
Let M stand for a Maths book and P stand for a Physics book.
Possibilities:

$$\begin{array}{ccccccccccc} 5M & \text{and} & 0P & \text{or} & 4M & \text{and} & 1P & \text{or} & 3M & \text{and} & 2P \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \binom{5}{8} & \times & \binom{6}{0} & + & \binom{8}{4} & \times & \binom{6}{1} & + & \binom{8}{3} & \times & \binom{6}{2} \end{array} \quad \text{(lower numbers in each case)}$$

$= 56 \times 1 + 70 \times 6 + 56 \times 15$
$= 56 + 420 + 840$
$= 1{,}316$

Exercise 5.2

Calculate:

1. $\binom{5}{2}$ 2. $\binom{8}{3}$ 3. $\binom{9}{5}$ 4. $\binom{10}{0}$ 5. $\binom{20}{18}$ 6. $\binom{30}{27}$

7. In how many ways can a committee of 4 people be chosen from 7 people?

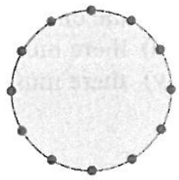
8. There are 15 pupils in a class. How many teams of 11 can be selected from the class? If one person in the class is made captain and must always be included in each team, how many teams can now be selected? If 2 pupils in the class refuse to play, how many teams can now be selected, if the captain must still be on every team?

9. In how many ways can a party of 6 children be chosen from a group of 10 children if:
   (i) any child may be selected?
   (ii) the oldest child must not be selected?
   (iii) the youngest child must be selected?
   (iv) the youngest and the oldest must both be selected?

10. A fifth-year student has to choose 4 subjects from the following list: Accounting, Biology, Chemistry, Physics, French, Applied Maths and Classical Studies.
   (i) How many different choices are possible?
   (ii) How many choices include French?
   (iii) How many choices do not include French?
   (iv) How many choices include Accounting and Biology?
   (v) How many choices include Applied Maths but not Chemistry?

11. Three delegates to form a committee are to be selected from eight members of a club. How many different committees can be formed if:
(i) there are no restrictions?
(ii) a certain member must be on each committee?
(iii) two particular members cannot both be on the committee?

12. Twelve distinct points are taken on the circumference of a circle (as shown).



(i) **(a)** Calculate the number of different chords that can be formed using these points as end points.
(b) How many different triangles can be formed using these points as vertices?
(ii) **(a)** Calculate the number of different quadrilaterals that can be formed using these points as vertices.
(b) Two of the ten points are labelled x and y respectively. How many of the above quadrilaterals have x and y as vertices?
(c) How many of the quadrilaterals do not have x and y as vertices?

13. From a set of six different coins, in how many ways can four or more coins be selected?

14. In how many ways can 12 different objects be divided into groups of 6, 4 and 2?

15. 5 Irishmen, 3 Frenchmen and 4 Germans are available for selection to a European committee of 4. If each nation has to be represented on each committee, in how many ways can the committee be selected?

16. **(i)** Find the number of different selections of 4 letters that can be made from the letters of the word *SPHERICAL*.
(ii) How many of these selections do not contain a vowel?
(iii) How many of these selections contain at least one vowel?

17. Find the number of different selections of 5 letters that can be made from the letters of the word *CHEMISTRY*.
How many of these selections contain at least one vowel?

18. A team of 6 players is to be chosen from a group of 10 players. One of the 6 is then to be elected as captain and another as vice-captain. In how many ways can this be done? (Hint: Select and then arrange.)

19. In how many ways can a committee of 7 people be selected from 4 men and 6 women, if the committee must have at least 4 women on it?

20. A committee of six is to be formed from eight students and five teachers. How many different committees can be formed if there are to be more teachers than students?

21. An examination consists of ten questions, four in section A and the remainder in section B. A candidate must attempt 5 questions, at least two of which must be from each section.
In how many different ways may the candidate select the five questions?

22. In how many ways can a committee of six be selected from five men and four women, if each committee consists of:
(i) an equal number of men and women?
(ii) at least three men?

23. A team of 5 players is to be chosen from 6 boys and 5 girls. If there must be more boys than girls, how many different teams can be formed?

24. A group consists of 5 men and 7 women. A committee of 4 must be chosen from the group. How many committees can be chosen in which there are an odd number of men?

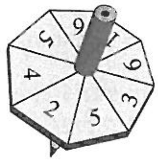
25. A club has only 5 women and 4 men as members. A team of 3 is to be chosen to represent the club. In how many ways can this be done if:
(i) there are no restrictions?
(ii) the club captain must be on the team?
(iii) there must be at least one woman on the team?
(iv) there must be more women than men on the team?

Probability

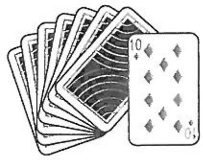
Probability involves the study of the laws of chance. It is a measure of the chance, or likelihood, of something happening.

toss coins

throw dice



spin a spinner



draw a card

If you carry out an operation, or experiment, using coins, dice, spinners or cards, then each toss, throw, spin or draw is called a **trial**.
The possible things that can happen from a trial are called **outcomes**. The outcomes of interest are called an **event**. In other words, an event is the set of successful outcomes.
For example, if you throw a die and you are interested in the probability of an even number, then the event is 2, 4, 6, the successful outcomes.
If E is an event, then $P(E)$ stands for the probability that the event occurs.
$P(E)$ is read 'the probability of E'.

Definition

The measure of the probability of an event, E, is given by:

$$P(E) = \frac{\text{number of successful outcomes}}{\text{number of possible outcomes}}.$$

The probability of an event is a number between 0 and 1, including 0 and 1.

$$\boxed{0 \leqslant P(E) \leqslant 1}$$

The value of $P(E)$ can be given as a fraction, decimal or percentage.

Note: $P(E) = \mathbf{0}$ means that an event is **impossible**.
$P(E) = \mathbf{1}$ means that an event is **certain**.

The set of all possible outcomes is called the '**sample space**'.
For example, the sample space for a normal die is 1, 2, 3, 4, 5 and 6.

Probability of an Event not Happening

If E is any event, then 'not E' is the event that E does not occur. Clearly E and 'not E' cannot occur at the same time. Either E or not E must occur. Thus, we have the following relationship between the probabilities of E and not E:

$$P(E) + P(\text{not } E) = 1$$
or
$$P(\text{not } E) = 1 - P(E)$$

Notes: **1.** It is very important **not** to count an outcome twice in an event when calculating probabilities.
2. In questions on probability, objects which are identical are treated as different.
3. The phrase '**drawn at random**' means each object is **equally likely** to be picked.
'**Unbiased**' means '**fair**'. '**Biased**' means '**unfair**' in some way.

Example

A bag contains 8 red, 3 blue and 13 yellow discs. A disc is selected at random from the bag. What is the probability that the disc selected is:
(i) red **(ii)** blue **(iii)** yellow **(iv)** not yellow?

Solution:

There are $8 + 3 + 13 = 24$ discs in the bag.

(i) $P(\text{red disc}) = \dfrac{\text{number of red discs}}{\text{total number of discs}} = \dfrac{8}{24} = \dfrac{1}{3}$

(ii) $P(\text{blue disc}) = \dfrac{\text{number of blue discs}}{\text{total number of discs}} = \dfrac{3}{24} = \dfrac{1}{8}$

(iii) $P(\text{yellow disc}) = \dfrac{\text{number of yellow discs}}{\text{total number of discs}} = \dfrac{13}{24}$

(iv) We are certain that the disc selected is yellow or not yellow.

$$\therefore \quad P(\text{yellow disc}) + P(\text{not a yellow disc}) = 1$$
$$P(\text{not a yellow disc}) = 1 - P(\text{yellow disc})$$
$$= 1 - \frac{13}{24} = \frac{11}{24}$$

Alternatively:

$$P(\text{not a yellow disc}) = \frac{\text{number of non-yellow discs}}{\text{total number of discs}} = \frac{11}{24}$$

(number of non-yellow discs = number of red discs + number of blue discs = $8 + 3 = 11$)

A pack of cards consists of 52 cards divided into four suits: Clubs (black), Diamonds (red), Hearts (red) and Spades (black). Each suit consists of 13 cards bearing the following values: 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King and Ace. The Jack, Queen and King are called 'picture cards'. So the total number of outcomes if one card is picked is 52.

Example

A card is drawn at random from a normal pack of 52 playing cards.
What is the probability that the card will be:
(i) an ace **(ii)** a spade **(iii)** black **(iv)** odd-numbered?

Solution:

(i) $P(\text{ace}) = \dfrac{\text{number of aces}}{\text{number of cards}} = \dfrac{4}{52} = \dfrac{1}{13}$

(ii) $P(\text{spade}) = \dfrac{\text{number of spades}}{\text{number of cards}} = \dfrac{13}{52} = \dfrac{1}{4}$

(iii) $P(\text{black card}) = \dfrac{\text{number of black cards}}{\text{number of cards}} = \dfrac{26}{52} = \dfrac{1}{2}$

(iv) Each suit has four odd numbers, 3, 5, 7 and 9. There are four suits.
Therefore, there are 16 cards with an odd number.

$P(\text{odd-numbered card}) = \dfrac{\text{number of cards with an odd number}}{\text{number of cards}} = \dfrac{16}{52} = \dfrac{4}{13}$

Conditional Probability

With conditional probability we are given some prior knowledge, or some extra condition, about the outcome. This usually reduces the size of the sample space. Consider parts **(iv)** and **(v)** of the next example.

Example

In a class, there are 21 boys and 15 girls. Three boys wear glasses and five girls wear glasses.
A pupil is picked at random from the class.

(i) What is the probability that the pupil is a boy?
(ii) What is the probability that the pupil wears glasses?
(iii) What is the probability that the pupil is a boy who wears glasses?

A girl is picked at random from the class.
(iv) What is the probability that she wears glasses?

A pupil wearing glasses is picked at random from the class.
(v) What is the probability that it is a boy?

Solution:

It is good practice to represent the information in a table (including the totals for each column and row).

	Boy	Girl	Total
Does not wear glasses	18	10	28
Wears glasses	3	5	8
Total	21	15	36

There are $21 + 15 = 36$ pupils in the class.

(i) $P(\text{boy}) = \dfrac{\text{number of boys}}{\text{number of pupils in the class}} = \dfrac{21}{36} = \dfrac{7}{12}$

(ii) $P(\text{pupil wears glasses}) = \dfrac{\text{number of pupils who wear glasses}}{\text{number of pupils in the class}} = \dfrac{8}{36} = \dfrac{2}{9}$

(iii) $P(\text{boy who wears glasses}) = \dfrac{\text{number of boys who wear glasses}}{\text{number of pupils in the class}} = \dfrac{3}{36} = \dfrac{1}{12}$

The next two questions require the use of conditional probability, where the size of the sample space has been reduced.

(iv) We are certain that the pupil picked is a girl. There are 15 girls in the class. 5 of these wear glasses.

P(when a girl is picked she wears glasses)

$$= \frac{\text{number of girls in the class who wear glasses}}{\text{number of girls in the class}} = \frac{5}{15} = \frac{1}{3}$$

(v) We are certain that the pupil picked wears glasses. There are 8 pupils who wear glasses. 3 of these pupils are boys.

P(when a pupil who wears glasses is picked, the pupil is a boy)

$$= \frac{\text{number of boys in the class who wear glasses}}{\text{number of pupils in the class who wear glasses}} = \frac{3}{8}$$

Combining Two Events

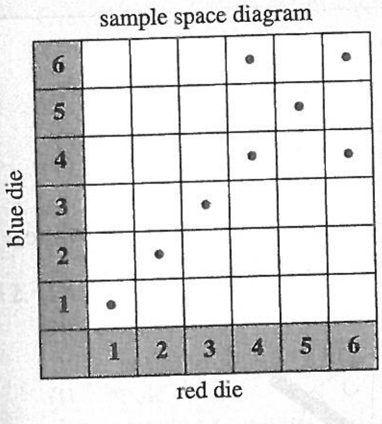
There are many situations where we have to consider two outcomes. In these situations all the possible outcomes, the **sample space**, can be listed in a sample space diagram (often called a '**two-way table**').

Example

Two dice, one red and the other blue, are thrown. What is the probability of getting two equal scores or of the scores adding up to 10?

Solution:

sample space diagram



36 possible outcomes (6×6)

The dots indicate where the two scores are equal and/or they add up to 10.

There are 8 dots.

$\therefore$ $P(\text{two equal scores or a total of 10}) = \dfrac{8}{36} = \dfrac{2}{9}$

Note: (5, 5) is **not** counted twice.

Note: Be careful if after the first selection there is no replacement, as in the next example.

Example

Of five balls in a bag, one bears the number 1, another the number 3, two others the number 4 and one the number 6. Two balls are drawn together. If an outcome is the product of the numbers on the two balls, write out the probability for each of the possible outcomes.

Solution:

Note: 'Product' means 'multiply'.

Picking two numbers at a time is the same as picking one after another without replacement.

sample space diagram

second selection					
⑥	6	18	24	24	
④	4	12	16		24
④	4	12		16	24
③	3		12	12	18
①		3	4	4	6
	①	③	④	④	⑥

first selection

The shaded regions indicate that you cannot pick the same ball twice.
From the diagram:

$P(3) = \frac{2}{20} = \frac{1}{10}$ $\quad P(4) = \frac{4}{20} = \frac{1}{5}$ $\quad P(6) = \frac{2}{20} = \frac{1}{10}$

$P(12) = \frac{4}{20} = \frac{1}{5}$ $\quad P(16) = \frac{2}{20} = \frac{1}{10}$ $\quad P(18) = \frac{2}{20} = \frac{1}{10}$

$P(24) = \frac{4}{20} = \frac{1}{5}$

Exercise 5.3

1. A box contains 36 coloured balls. 12 are red, 15 are blue, 3 are yellow and the rest are white. One ball is selected at random from the box. Calculate the probability of selecting a:
   (i) red ball **(ii)** blue ball **(iii)** yellow ball **(iv)** white ball.

2. A fair spinner has eight sides as shown.
   The sides are labelled A, B, B, C, C, C, C and F.
   The spinner is spun once.
   What is the probability that the spinner lands on:
   (i) A **(ii)** B **(iii)** C?