CHAPTER 14

INTEGRATION

The Indefinite Integral

> Integration is the **reverse** process of differentiation.

The process of finding a function from its derivative is called '**integration**'.
For example, we know that if $f(x) = x^2$, then $f'(x) = 2x$.
Now suppose that we are given $f'(x) = 2x$ and asked to find $f(x)$. In other words, we start with the d
ative and work '***backwards***' to the original function.
However, if $f(x) = x^2 + 10$, $f'(x) = 2x$ and if $f(x) = x^2 - 3$, $f'(x) = 2x$.
In fact, if $f(x) = x^2 + c$, then $f'(x) = 2x$, where c is a constant.
In other words, we do not know whether the original function contained a constant term or not.

Notation:

$$\int f'(x)\,\mathrm{d}x = f(x) + c$$

The symbol for integration is $\int$, an elongated s.

$\mathrm{d}x$ indicates that the integration is with respect to the variable x.
c is called the 'constant of integration'.

Note: $\int 2x\,\mathrm{d}x$ is read as 'the integral of $2x\,\mathrm{d}x$' or 'the integral of $2x$ with respect to x'.

Basic rule:

$$\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + c \qquad (n \neq -1)$$

In words: Increase the power by 1 and divide by the new power.
If a is a constant, $\int ax^n\,\mathrm{d}x = a\int x^n\,\mathrm{d}x = a\,\frac{x^{n+1}}{n+1} + c$, $(n \neq -1)$.
A constant factor of the integrand can be taken outside the symbol of integration.

$$\int (f(x) \pm g(x))\,\mathrm{d}x = \int f(x)\,\mathrm{d}x \pm \int g(x)\,\mathrm{d}x.$$

To integrate a sum or difference, add or subtract the individual integrals.

Note: Before integrating, **all** terms must be written in the form x^n or ax^n, where a is a constant.

There is no 'product', 'quotient' or 'chain' rule in integration.
It is possible to check your answer in integration by differentiating your answer and seeing whether you get back to the original integral.

Example

Find: **(i)** $\int 3x^2\,dx$ **(ii)** $\int \frac{1}{x^3}\,dx$ **(iii)** $\int \frac{1}{\sqrt{x}}\,dx$

Solution:

(i)
$$\int 3x^2\,dx$$
$$= \frac{3x^3}{3} + c$$
$$= x^3 + c$$

(ii)
$$\int \frac{1}{x^3}\,dx$$
$$= \int x^{-3}\,dx$$
$$= \frac{x^{-2}}{-2} + c$$
$$= -\frac{1}{2x^2} + c$$

(iii)
$$\int \frac{1}{\sqrt{x}}\,dx$$
$$= \int x^{-1/2}\,dx$$
$$= \frac{x^{1/2}}{\frac{1}{2}} + c$$
$$= 2x^{1/2} + c \text{ or } 2\sqrt{x} + c$$

Sometimes it is necessary to manipulate the integrand to write each part in the form ax^n.

Example

Find: **(i)** $\int \left(x + \frac{1}{x}\right)^2 dx$ **(ii)** $\int \frac{3+x}{\sqrt{x}}\,dx$

Solution

(i)
$$\int \left(x + \frac{1}{x}\right)^2 dx$$
$$= \int \left(x^2 + 2 + \frac{1}{x^2}\right) dx$$
$$= \int (x^2 + 2 + x^{-2})\,dx$$
$$= \frac{x^3}{3} + 2x + \frac{x^{-1}}{-1} + c$$
$$= \tfrac{1}{3}x^3 + 2x - \frac{1}{x} + c$$

(ii)
$$\int \left(\frac{3+x}{\sqrt{x}}\right) dx$$
$$= \int \left(\frac{3}{x^{1/2}} + \frac{x^1}{x^{1/2}}\right) dx \qquad \left(\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}\right)$$
$$= \int (3x^{-1/2} + x^{1/2})\,dx$$
$$= \frac{3x^{1/2}}{\frac{1}{2}} + \frac{x^{3/2}}{\frac{3}{2}} + c$$
$$= 6x^{1/2} + \tfrac{2}{3}x^{3/2} + c$$

Sometimes if we are given some extra information, we may be asked to find the constant of integra
or an expression for $f(x)$.

Example

(i) Find the constant of integration, given that $\int(3x^2+1)\,dx = 6$ when $x = 2$.

(ii) Find the function $y = f(x)$, given that $f'(x) = 5 - 2x$ and that the graph of $y = f(x)$ passe
through the point (1, 7).

Solution:

(i) $\int(3x^2+1)\,dx$

$= \frac{3x^3}{3} + x + c$

$= x^3 + x + c$

Given: this is = 6 when $x = 2$.

$\therefore\quad (2)^3 + (2) + c = 6$

$8 + 2 + c = 6$

$10 + c = 6$

$c = -4$

Thus, the constant of integration is -4.

(ii) **Given**: $f'(x) = 5 - 2x$

$\int f'(x)\mathrm{d}x = \int(5-2x)\,\mathrm{d}x$

(integrate both sides with respect to x)

$f(x) = 5x - x^2 + c$

Given: $f(1) = 7$ (or $y = 7$ when $x = 1$

$\therefore\quad 5(1) - (1)^2 + c = 7$

$5 - 1 + c = 7$

$4 + c = 7$

$c = 3$

Thus, $f(x) = 3 + 5x - x^2$

Exercise 14.1

Find:

1. $\int x^3\,dx$ **2.** $\int x^2\,dx$ **3.** $\int 5x^4\,dx$ **4.** $\int -x\,dx$ **5.** $\int -2x^2$ d

6. $\int 5\,dx$ **7.** $\int -2\,dx$ **8.** $\int \frac{1}{x^2}\,dx$ **9.** $\int \sqrt{x}\,dx$ **10.** $\int \frac{2}{\sqrt{x}}\,dx$

11. $\int(3x^2+8x)\,dx$ **12.** $\int(x^2+2x)\,dx$ **13.** $\int(2x^2-5x)$ (

14. $\int\left(x^2-\frac{1}{x^2}\right)dx$ **15.** $\int\left(4x^3-\frac{2}{x^3}\right)dx$ **16.** $\int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)$ c

17. $\int x(2+x)\mathrm{d}x$ **18.** $\int x^2(x+5)\,dx$ **19.** $\int\sqrt{x}(x+1)$ d

20. $\int\left(\frac{x^2+3}{x^2}\right)dx$ **21.** $\int\left(\frac{x+1}{x^3}\right)dx$ **22.** $\int\left(\frac{3x^2-5}{\sqrt{x}}\right)$ d

23. $\int\left(\frac{x^4-2x^3+x^2}{x}\right)dx$ **24.** $\int\left(x-\frac{1}{x}\right)^2 dx$ **25.** $\int\left(x^2+\frac{1}{x}\right)^2$ d

26. If $f'(x) = 3x^2 - 2x$, and $f(2) = 9$, find $f(x)$.

27. If $f'(x) = 3x^2 - \sqrt{x}$ and $f(0) = 4$, find $f(x)$.

28. If $f'(t) = 4t^3 - 6t$ and $f(-2) = 8$, find $f(t)$.

29. A curve contains the point $(\frac{1}{2}, 3)$ and its slope at any point (x, y) on the curve is given by $f'(x) = 16x^3 + 2x + 1$. Find $f(x)$.

30. A curve contains the point $(1, 7)$ and its slope at any point (x, y) on the curve is given by $\dfrac{dy}{dx} = 3x^2 + 4$. Find the equation of the curve.

The Definite Integral

$$\boxed{\int_a^b f'(x)\,dx = \big[f(x)\big]_a^b = f(b) - f(a)}$$

We call $\int_a^b f'(x)\,dx$ a **definite integral**, as it gives a definite answer.

The dx indicates that the limits a and b are x limits.
The constant a is called the **lower limit** of the integral.
The constant b is called **upper limit** of the integral.
The is no constant of integration when we evaluate definite integrals, as they cancel each other out.

$$\int_a^b f'(x)\,dx = -\int_b^a f'(x)\,dx$$

If the limits are swopped, the sign of the definite integral is changed.
Definite integrals can be used to find the area beneath a curve and volumes of rotation.

Example

Evaluate: **(i)** $\int_1^3 (x^2+2x)\,dx$ **(ii)** $\int_1^4 \left(\sqrt{x}-\frac{3}{x}\right)^2 dx$

Solution:

(i) $\int_1^3 (x^2+2x)\,dx$

$= \left[\frac{x^3}{3}+x^2\right]_1^3$

$= \left[\frac{(3)^3}{3}+(3)^2\right]-\left[\frac{(1)^3}{3}+(1)^2\right]$

$= [\frac{27}{3}+9]-[\frac{1}{3}+1]$

$= 18-\frac{4}{3}$

$= 16\frac{2}{3}$

(ii) $\int_1^4 \left(\sqrt{x}-\frac{3}{x}\right)^2 dx = \int_1^4 \left(x-\frac{6}{\sqrt{x}}+\frac{9}{x^2}\right) dx$

$= \int_1^4 (x-6x^{-1/2}+9x^{-2})\,dx$

$= \left[\frac{x^2}{2}-\frac{6x^{1/2}}{\frac{1}{2}}+\frac{9x^{-1}}{-1}\right]_1^4$

$= \left[\frac{x^2}{2}-12\sqrt{x}-\frac{9}{x}\right]_1^4$

$= \left[\frac{4^2}{2}-12\sqrt{4}-\frac{9}{4}\right]-\left[\frac{1^2}{2}-12\sqrt{1}-\frac{9}{1}\right]$

$= [8-24-2\frac{1}{4}]-[\frac{1}{2}-12-9]$

$= -18\frac{1}{4}+20\frac{1}{2}=2\frac{1}{4}$

Exercise 14.2

Evaluate each of the following definite integrals:

1. $\int_1^2 3x^2\,dx$ **2.** $\int_0^2 4x^3\,dx$ **3.** $\int_1^2 x^3\,dx$ **4.** $\int_1^4 \sqrt{x}\,dx$

5. $\int_0^4 (4x-x^2)\,dx$ **6.** $\int_1^3 x(2-3x)\,dx$ **7.** $\int_{-2}^2 x(x+4)\,dx$ **8.** $\int_2^3 \frac{1}{x^2}\,dx$

9. $\int_{-1}^1 \frac{1}{x^3}\,dx$ **10.** $\int_2^4 \left(\frac{1}{x^2}+3\right) dx$ **11.** $\int_4^9 \left(1-\frac{3}{\sqrt{x}}\right) dx$ **12.** $\int_1^4 \left(3-\frac{1}{\sqrt{x}}\right) dx$

13. $\int_1^2 \left(x+\frac{1}{x}\right)^2 dx$ **14.** $\int_1^{16} \left(\frac{\sqrt{x}-4}{\sqrt{x}}\right) dx$ **15.** $\int_1^2 \frac{x^2+x}{x^4}\,dx$ **16.** $\int_1^4 \frac{3x-2\sqrt{x}}{x}\,dx$

17. Express $\frac{4x^2-9}{4x-6}$ in the form $\frac{1}{a}(ax+b)$ and, hence, evaluate $\int_0^2 \frac{4x^2-9}{4x-6}\,dx$.

18. Express $\frac{x^3-8}{x-2}$ in the form x^2+px+q and, hence, evaluate $\int_0^1 \frac{x^3-8}{x-2}\,dx$.

In each of the following find the value of $k > 0$:

19. $\int_0^k x^2 \, \mathrm{d}x = 9$ **20.** $\int_1^k (2x+3) \, \mathrm{d}x = 6$ **21.** $\int_0^9 \frac{k}{\sqrt{x}} \, \mathrm{d}x = 30$

22. Verify that $\int_{4-k}^{4+k} (x-4) \, \mathrm{d}x = 0$

Integration by Substitution

Some integrals may be found more easily by using substitution. When evaluating an indefinite integral by substitution the answer **must be transformed back to a function of the original variable.**

Example

Find: (i) $\int 2x(x^2+1)^4 \, \mathrm{d}x$ (ii) $\int x\sqrt{1-x^2} \, \mathrm{d}x$

Solution:

(i) $\int 2x(x^2+1)^4 \, \mathrm{d}x$

$u = x^2 + 1$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

Substitution

let $u = x^2 + 1$

$\mathrm{d}u = 2x \, \mathrm{d}x$

$$\int 2x(x^2+1)^4 \, \mathrm{d}\cancel{x}$$

$$= \int u^4 \, \mathrm{d}u$$

$$= \frac{u^5}{5} + c$$

$$= \frac{(x^2+1)^5}{5} + c$$

(ii) $\int x\sqrt{1-x^2} \, \mathrm{d}x$

$u = 1 - x^2$

$\frac{du}{dx} = -2x$

$du = -2x\,dx$

Substitution

let $u = 1 - x^2$

$\mathrm{d}u = -2x \, \mathrm{d}x$

$-\frac{1}{2} \, \mathrm{d}u = x \, \mathrm{d}x$

$$\int x\sqrt{1-x^2} \, \mathrm{d}x$$

$$= -\tfrac{1}{2} \int u^{1/2} \, \mathrm{d}u$$

$$= -\tfrac{1}{2} \cdot \frac{u^{3/2}}{\frac{3}{2}} + c$$

$$= -\tfrac{1}{2} \cdot \tfrac{2}{3} u^{3/2} + c$$

$$= -\tfrac{1}{3} u^{3/2} + c$$

$$= -\tfrac{1}{3}(1-x^2)^{3/2} + c$$

Sometimes we have to rearrange as well as substitute.

Example

Evaluate: $\displaystyle\int_2^3 \frac{x}{\sqrt{x-1}}\,\mathrm{d}x$

Solution:

$$\int \frac{x}{\sqrt{x-1}}\,\mathrm{d}x$$

$$= \int \left(\frac{u+1}{u^{1/2}}\right)\mathrm{d}u = \int (u^{1/2} + u^{-1/2})\,\mathrm{d}u$$

$$= \frac{u^{3/2}}{\frac{3}{2}} + \frac{u^{1/2}}{\frac{1}{2}}$$

$$= \tfrac{2}{3}u^{3/2} + 2u^{1/2}$$

$$= \left[\tfrac{2}{3}(x-1)^{3/2} + 2(x-1)^{1/2}\right]$$

Substitution

let $u = x - 1$
$\mathrm{d}u = \mathrm{d}x$
We must also do some rearranging
$u = x - 1$
$u + 1 = x$

$$\therefore \int_1^3 \frac{x}{\sqrt{x-1}}\,\mathrm{d}x$$

$(2^{3/2} = 2^1 \cdot 2^{1/2} = 2\sqrt{2})$

$$= \left[\tfrac{2}{3}(x-1)^{3/2} + 2(x-1)^{1/2}\right]_1^3$$

$$= \left[\tfrac{2}{3}(2)^{3/2} + 2(2)^{1/2}\right] - \left[\tfrac{2}{3}(1-1)^{3/2} + 2(1-1)^{1/2}\right] = \tfrac{2}{3}(2\sqrt{2}) + 2\sqrt{2} = 2\sqrt{2}(\tfrac{2}{3}+1) = 2\sqrt{2}\cdot\tfrac{5}{3} = \frac{10\sqrt{2}}{3}$$

However, rather than remain with the limits for x, which necessitates expressing the integral in terms of x, we may change the limits to those of u.

Change of limits

$u = x - 1$

$x = 3$	$x = 1$
$u = 3 - 1$	$u = 1 - 1$
$u = 2$	$u = 0$

$$\therefore \int_1^3 \frac{x}{\sqrt{x-1}}\,\mathrm{d}x$$

$$= \int_0^2 \frac{u+1}{u^{1/2}}\,\mathrm{d}u$$

$$= \left[\tfrac{2}{3}u^{3/2} + 2u^{1/2}\right]_0^2$$

$$= \left[\tfrac{2}{3}(2)^{3/2} + 2(2)^{1/2}\right] - \left[\tfrac{2}{3}(0)^{3/2} + 2(0)^{1/2}\right] = \tfrac{2}{3}(2\sqrt{2}) + 2(\sqrt{2}) = 2\sqrt{2}(\tfrac{2}{3}+1) = 2\sqrt{2}\cdot\tfrac{5}{3} = \frac{10\sqrt{2}}{3}$$

To change the limits to those of u, or not, is a matter of personal choice. However, it often turns th the substitution is easier when the limits are changed to those of u.

Choosing the substitution is a skill most students take a long time to master. However, do not be afrai to try a substitution. If you use a wrong one, you will soon find out. So go back to the start and try different substitution.

Exercise 14.3

Find the following integrals, in each case using the suggested substitution:

1. $\int (x+1)^4\,dx \quad (u = x+1)$
2. $\int 2x(x^2-4)^3\,dx \quad (u = x^2-4)$
3. $\int 4x(2x^2-3)^3\,dx \quad (u = 2x^2-3)$
4. $\int 3x^2(x^3+1)^5\,dx \quad (u = x^3+1)$
5. $\int \sqrt{4x+3}\,dx \quad (u = 4x+3)$
6. $\int 2x\sqrt{x^2+5}\,dx \quad (u = x^2+5)$
7. $\int x\sqrt{x^2-3}\,dx \quad (u = x^2-3)$
8. $\int \frac{3x^2+2}{(x^3+2x)^6}\,dx \quad (u = x^3+2x)$
9. $\int x^2\sqrt{x^3-2}\,dx \quad (u = x^3-2)$
10. $\int \frac{x}{\sqrt{x-3}}\,dx \quad (u = x-3)$

Find each of the following:

11. $\int x(x^2+1)^5\,dx$
12. $\int \frac{x}{\sqrt{x^2+3}}\,dx$
13. $\int \frac{4x}{(1+x^2)^3}\,dx$
14. $\int x(x+3)^5\,dx$
15. $\int x(x-2)^4\,dx$
16. $\int \frac{x}{\sqrt{x+4}}\,dx$

Evaluate each of the following:

17. $\int_1^2 2x(x^2+1)^3\,dx$
18. $\int_1^2 x^2(x^3+1)^3\,dx$
19. $\int_0^1 \frac{x^2}{(x^3+1)^4}\,dx$
20. $\int_0^1 x(1+x^2)^5\,dx$
21. $\int_3^{\sqrt{14}} x\sqrt{x^2-5}\,dx$
22. $\int_1^{\sqrt{6}} \frac{2x}{\sqrt{x^2+3}}\,dx$
23. Verify that (i) $\int_3^4 x\sqrt{25-x^2}\,dx = \frac{37}{3}$ (ii) $\int_0^{\sqrt{3}} \frac{x}{\sqrt{x^2+1}}\,dx = 1$

Integrating Trigonometric Functions

$$\int \cos(nx+k)\,dx = \frac{\sin(nx+k)}{n} + c$$

$$\int \sin(nx+k)\,dx = -\frac{\cos(nx+k)}{n} + c$$

These integrals can be written down directly without substitution.

Note: When integrating the angle **must** be in radians.

Example

Find: **(i)** $\int \cos 3x\, dx$ **(ii)** $\int \sin(5x+2)\, dx$ **(iii)** $\int \left[\cos 2\theta + \sin\left(4\theta - \frac{\pi}{2}\right)\right] d\theta$

Solution:

(i) $\int \cos 3x\, dx = \frac{\sin 3x}{3} + c$

(ii) $\int \sin(5x+2)\, dx = -\frac{\cos(5x+2)}{5} + c$

(iii) $\int \left[\cos 2\theta + \sin\left(4\theta - \frac{\pi}{2}\right)\right] d\theta = \frac{\sin 2\theta}{2} - \frac{\cos\left(4\theta - \frac{\pi}{2}\right)}{4} + c$

Products $\int \sin mx \,.\, \cos nx$

Always write the bigger angle first and then use the formulae on page 9 of the tables to change product into a sum or difference.

Example

Evaluate: $\int_0^{\pi/6} \cos 2\theta \sin 4\theta \, d\theta$

Solution:

$$\int_0^{\pi/6} \cos 2\theta \sin 4\theta \, d\theta$$

$$= \int_0^{\pi/6} \sin 4\theta \cos 2\theta \, d\theta \quad \text{[put bigger angle first]}$$

$$= \int_0^{\pi/6} \tfrac{1}{2}\left[\sin(4\theta + 2\theta) + \sin(4\theta - 2\theta)\right] d\theta$$

(from page 9, $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-\beta])$)

$$= \frac{1}{2}\int_0^{\pi/6} (\sin 6\theta + \sin 2\theta)\, d\theta$$

$$= \frac{1}{2}\left[-\frac{\cos 6\theta}{6} - \frac{\cos 2\theta}{2}\right]_0^{\pi/6}$$

$$= -\frac{1}{2}\left[\frac{\cos 6\theta}{6} + \frac{\cos 2\theta}{2}\right]_0^{\pi/6}$$

$$= -\frac{1}{2}\left[\left(\frac{\cos \pi}{6} + \frac{\cos \frac{\pi}{3}}{2}\right) - \left(\frac{\cos 0}{6} + \frac{\cos 0}{2}\right)\right]$$

$$= -\tfrac{1}{2}[(-\tfrac{1}{6} + \tfrac{1}{4}) - (\tfrac{1}{6} + \tfrac{1}{2})]$$

$$= -\tfrac{1}{2}(-\tfrac{7}{12}) = \tfrac{7}{24}$$

Even powers of cos x and sin x

To integrate even powers of cos x and sin x, rewrite the integrand using the double angle identities on page 9 of the tables. Find:

$$\cos^2 A = \tfrac{1}{2}(1 + \cos 2A) \qquad \text{and} \qquad \sin^2 A = \tfrac{1}{2}(1 - \cos 2A)$$

Example

Evaluate: **(i)** $\displaystyle\int_0^{\pi/4} \sin^2 x \, dx$ **(ii)** $\displaystyle\int_{\pi/8}^{\pi/4} 2\cos^2 4\theta \, d\theta$

Solution:

(i) $$\int_0^{\pi/4} \sin^2 x \, dx$$

$$= \int_0^{\pi/4} \tfrac{1}{2}(1 - \cos 2x) \, dx$$

$$= \frac{1}{2}\int_0^{\pi/4} (1 - \cos 2x) \, dx$$

$$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left(\left[\frac{\pi}{4} - \frac{\sin\frac{\pi}{2}}{2}\right] - \left[0 - \frac{\sin 0}{2}\right]\right)$$

$$= \frac{1}{2}\left(\left[\frac{\pi}{4} - \frac{1}{2}\right] - [0 - 0]\right)$$

$$= \frac{1}{2}\left[\frac{\pi}{4} - \frac{1}{2}\right] = \frac{\pi}{8} - \frac{1}{4}$$

(ii) $$\int_{\pi/8}^{\pi/4} 2\cos^2 4\theta \, d\theta$$

$$= \int_{\pi/8}^{\pi/4} 2 \cdot \tfrac{1}{2}(1 + \cos 8\theta) \, d\theta$$

$$= \int_{\pi/8}^{\pi/4} (1 - \cos 8\theta) \, d\theta$$

$$= \left[\theta + \frac{\sin 8\theta}{8}\right]_{\pi/8}^{\pi/4}$$

$$= \left[\frac{\pi}{4} + \frac{\sin 2\pi}{8}\right] - \left[\frac{\pi}{8} + \frac{\sin \pi}{8}\right]$$

$$= \left[\frac{\pi}{4} + 0\right] - \left[\frac{\pi}{8} + 0\right]$$

$$= \frac{\pi}{4} - \frac{\pi}{8} = \frac{\pi}{8}$$

Exercise 14.4

1. $\displaystyle\int \cos 2x \, dx$ **2.** $\displaystyle\int \sin 4x \, dx$ **3.** $\displaystyle 3\int \cos 3x \, dx$

4. $\displaystyle\int \sin(2x + 3) \, dx$ **5.** $\displaystyle\int \cos(5x + 4) \, dx$ **6.** $\displaystyle\int \sin\left(8x - \frac{\pi}{4}\right) dx$

7. $\displaystyle\int (\cos 4\theta - \sin 2\theta) \, d\theta$ **8.** $\displaystyle 8\int (\cos 2\theta - \sin 8\theta) \, d\theta$ **9.** $\displaystyle\int 2\sin 4x \cos 2x \, dx$

10. $\displaystyle\int 2\cos 6x \sin x \, dx$ **11.** $\displaystyle\int \sin 3x \sin 5x \, dx$ **12.** $\displaystyle\int \cos 2x \cos 3x \, dx$

13. $\displaystyle\int \cos^2 x \, dx$ **14.** $\displaystyle\int 2\sin^2 2\theta \, d\theta$ **15.** $\displaystyle\int 4\sin^2 3x \, dx$

16. $\displaystyle\int \cos^2 5\theta \, d\theta$

Evaluate each of the following:

17. $\int_0^{\pi/4} \cos\theta \, d\theta$

18. $\int_0^{\pi/2} \cos 2x \, dx$

19. $3\int_0^{\pi/4} \sin 4x \, dx$

20. $\int_0^{\pi/2} \cos 3x \cos 2x \, dx$

21. $\int_0^{\pi/6} \cos 2x \sin 4x \, dx$

22. $\int_0^{\pi/4} \cos^2 x \, dx$

23. $\int_0^{\pi/6} \cos^2 3x \, dx$

24. $2\int_{\pi/8}^{\pi/4} 2\sin^2 4\theta \, d\theta$

For integrals of the form $\int \cos^2 x \sin x \, dx$, $\int \sin^2 x \cos x \, dx$ or $\int \cos^4 x \sin x \, dx$:

the substitution required is let u = (trigonometric function with the even power).

25. Evaluate: **(i)** $\int_0^{\pi/2} \cos^2 x \sin x \, dx$ **(ii)** $\int_0^{\pi/2} \cos\theta(2 + 3\sin^2\theta) \, d\theta$

26. Find $\int \sin^2 x \cos x \, dx$. Hence, or otherwise, evaluate $\int_0^{\pi/2} \cos^3 x \, dx$.
(**Hint:** Let $\cos^3 x = \cos^2 x \cos x = (1 - \sin^2 x)\cos x$.)

Integration of Exponential Functions

$$\boxed{\int e^{ax} \, dx = \frac{e^{ax}}{a} + c \qquad \text{or} \qquad \int e^{ax+b} \, dx = \frac{e^{ax+b}}{a} + c}$$

These integrals can be written down directly without substitution.
However, if the power of e is not ax or $ax + b$, we use the method of substitution.

$$\boxed{\text{The substitution is: let } u = (\text{power of } e)}$$

Example

(i) Find $\int e^{4x-3} \, dx$ **(ii)** Evaluate $\int_0^1 4xe^{x^2} \, dx$

Solution:

(i) $\int e^{4x-3} \, dx = \dfrac{e^{4x-3}}{4} + c$

(ii) $\int_0^1 4xe^{x^2} \, dx$

$$\int 4xe^{x^2} \, dx = 2\int e^u \, du$$
$$= 2e^u$$
$$= 2e^{x^2}$$

Substitution

$u = x^2$ (power of e)
$du = 2x \, dx$
$2 \, du = 4x \, dx$

$$\therefore \quad \int_0^1 4xe^{x^2} \, dx = 2\left[e^{x^2}\right]_0^1 = 2\left[e^1 - e^0\right] = 2(e - 1)$$

$\sin 4x \, dx$

$os^2 x \, dx$

Example

Evaluate: $\displaystyle\int_0^1 (x+2)e^{x(x+4)}\,dx$.

Solution:

$$\int (x+2)e^{x(x+4)}\,dx$$

$$= \int (x+2)e^{x^2+4x}\,dx$$

$$= \tfrac{1}{2}\int e^u\,du$$

$$= \tfrac{1}{2}e^u = \tfrac{1}{2}e^{x^2+4x}$$

$$\therefore \quad \int_0^1 (x+2)e^{x^2+4x}\,dx = \tfrac{1}{2}\Big[e^{x^2+4x}\Big]_0^1 = \tfrac{1}{2}(e^5 - e^0) = \tfrac{1}{2}(e^5 - 1)$$

Substitution

$$u = x^2 + 4x$$
$$du = (2x+4)\,dx$$
$$\tfrac{1}{2}\,du = (x+2)\,dx$$

Exercise 14.5

Find:

1. $\displaystyle\int e^{3x}\,dx$ **2.** $\displaystyle\int e^{2x+3}\,dx$ **3.** $\displaystyle\int e^{-4x}\,dx$ **4.** $\displaystyle\int e^{1-3x}\,dx$

5. $\displaystyle\int \frac{1}{e^{2x}}\,dx$ **6.** $\displaystyle 2\int e^{x/2}\,dx$ **7.** $\displaystyle\int (e^{2x}+e^x)\,dx$ **8.** $\displaystyle\int\left(e^{3x}-\frac{1}{e^{3x}}\right)dx$

9. $\displaystyle\int 3x^2e^{x^3}\,dx$ **10.** $\displaystyle\int (2x+3)e^{x^2+3x}\,dx$ **11.** $\displaystyle\int (1+\cos x)e^{x+\sin x}\,dx$

Evaluate each of the following:

12. $\displaystyle\int_0^1 e^{2x}\,dx$ **13.** $\displaystyle\int_{-1}^1 e^{3x-1}\,dx$ **14.** $\displaystyle\int_0^2 e^{4x-1}\,dx$

15. $\displaystyle\int_0^1 2xe^{x^2+1}\,dx$ **16.** $\displaystyle\int_0^2 \frac{e^{2x}-2}{e^x}\,dx$ **17.** $\displaystyle\int_2^3 (x-1)e^{x(x-2)}\,dx$

18. Evaluate $\displaystyle\int_0^{\pi/2} \cos x\, e^{\sin x}\,dx$. (Hint: let $u = \sin x$)

19. Evaluate $\displaystyle\int_0^1 \frac{e^x}{1+e^x}\,dx$. (Hint: let $u = 1 + e^x$)

20. Evaluate $\displaystyle\int_0^{\ln 2} \frac{e^x}{e^x+3}\,dx$.

21. Evaluate $\displaystyle\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$. $\left(\text{Hint: } \dfrac{e^{\sqrt{x}}}{\sqrt{x}} = \dfrac{1}{\sqrt{x}}\,e^{\sqrt{x}}\right)$

Integrals leading to a Logarithmic Function

$$\int \frac{1}{x}\,dx = \ln x + c \qquad \text{or} \qquad \int \frac{1}{ax+b}\,dx = \frac{\ln(ax+b)}{a} + c$$

These integrals can be written down directly without substitution.
However, in many cases we have to use the method of substitution and often, after the substit there is also some rearranging to do.

The substitution is: let u = (the bottom).

Notes: $\int \frac{5}{x}\,dx = \int 5\frac{1}{x}\,dx = 5\int \frac{1}{x}\,dx = 5\ln x + c$

$\int \frac{dx}{x} = \int \frac{1}{x}\,dx = \ln x + c$

$\int \frac{f'(x)}{f(x)} = \ln f(x) + c$

If the top is the derivative of the bottom, then the answer is ln (bottom) + c,

i.e. $\int \frac{\text{derivative of the bottom}}{\text{bottom}} = \ln(\text{bottom}) + c$

This can also be written down directly without substitution.

Example

Find: **(i)** $\int \frac{3}{5x+2}\,dx$ **(ii)** $\int \frac{2x}{x^2+3}\,dx$

Solution:

(i) $\int \frac{3}{5x+2}\,dx = 3\int \frac{1}{5x+2}\,dx$

Substitution

$u = 5x+2$ (bottom)
$du = 5\,dx$
$\frac{1}{5}\,du = dx$

$\therefore\ 3\int \frac{1}{5x+2}\,dx = 3\cdot\frac{1}{5}\int \frac{1}{u}\,du$
$= \frac{3}{5}\ln u + c$
$= \frac{3}{5}\ln(5x+2) + c$

(ii) $\int \frac{2x}{x^2+3}\,dx$

Substitution

$u = x^2+3$ (bottom)
$du = 2x\,dx$

$\therefore\ \int \frac{2x}{x^2+3}\,dx$
$= \int \frac{1}{u}\,du$
$= \ln u + c$
$= \ln(x^2+3) + c$

substitution,

Example

Evaluate: $\int_0^2 \frac{x}{x+2}\,dx$.

Solution:

$$\int \frac{x}{x+2}\,dx = \int \frac{u-2}{u}\,du$$
$$= \int\left(1-\frac{2}{u}\right)du$$
$$= \int\left(1-2\frac{1}{u}\right)du$$
$$= u - 2\ln u$$
$$= (x+2) - 2\ln(x+2)$$

Substitution

$$u = x+2$$
$$du = dx$$

We must also do some rearranging

$$u = x+2$$
$$u-2 = x$$

$$\therefore \quad \int_0^2 \frac{x}{x+2}\,dx = \Big[(x+2) - 2\ln(x+2)\Big]_0^2$$
$$= (4-2\ln 4) - (2-2\ln 2)$$
$$= 4 - 2\ln 4 - 2 + 2\ln 2$$
$$= 2 - 2(\ln 4 - \ln 2)$$
$$= 2 - 2\ln\left(\tfrac{4}{2}\right) = 2 - 2\ln 2$$

Exercise 14.6

Find:

1. $\int \frac{1}{x+2}\,dx$ **2.** $\int \frac{3}{x-5}\,dx$ **3.** $\int \frac{1}{2x}\,dx$ **4.** $\int \frac{4}{4x+5}\,dx$

5. $\int \frac{2x}{x^2+1}\,dx$ **6.** $\int \frac{8}{4x+3}\,dx$ **7.** $\int \frac{2}{1-2x}\,dx$ **8.** $\int \frac{x}{1+x^2}\,dx$

Evaluate each of the following:

9. $\int_1^2 \frac{1}{x}\,dx$ **10.** $\int_0^4 \frac{5}{x+1}\,dx$ **11.** $\int_1^2 \frac{3x^2}{x^3+1}\,dx$ **12.** $\int_3^4 \frac{2x-6}{x^2-6x+10}\,dx$

13. $\int_1^2 \frac{x^2+1}{x}\,dx = \frac{a}{b} + \ln b$, $a, b \in \mathbf{N}$. Evaluate $\sqrt{a^2+3b+1}$.

14. Evaluate: **(i)** $\int_0^6 \frac{x}{x+2}\,dx$ **(ii)** $\int_3^4 \frac{x}{x-2}\,dx$ **(iii)** $\int_1^3 \frac{x}{x+1}\,dx$

15. Evaluate: **(i)** $\ln\sqrt{e}$ **(ii)** $\int_{\sqrt{e}}^{e} \frac{1}{x}\,dx$

Integrals of the Form $\int \frac{dx}{\sqrt{a^2 - x^2}}$ and $\int \frac{dx}{a^2 + x^2}$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + c$$

and

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$$

These integrals are on page 41 of the tables and can be written down directly as long as the coefficient of x^2 is 1.

Example

Find: **(i)** $\int \frac{dx}{\sqrt{16 - x^2}}$ **(ii)** $\int \frac{dx}{9 + x^2}$

Solution:

(i)
$$\int \frac{dx}{\sqrt{16 - x^2}}$$
$$= \int \frac{dx}{\sqrt{4^2 - x^2}}$$
$$= \sin^{-1}\frac{x}{4} + c$$

(ii)
$$\int \frac{dx}{9 + x^2}$$
$$= \int \frac{dx}{3^2 + x^2}$$
$$= \tfrac{1}{3}\tan^{-1}\frac{x}{3} + c$$

When the coefficient of x^2 is not 1 we use a substitution. If we use u as the new variable, we arrange our substitution so that the coefficient of u^2 is 1.

Example

Evaluate: **(i)** $\int_0^{1/4} \frac{dx}{\sqrt{1 - 4x^2}}$ **(ii)** $\int_0^{2/3} \frac{dx}{4 + 9x^2}$

Solution:

(i) $$\int \frac{dx}{1+4x^2} = \int \frac{dx}{\sqrt{1^2-(2x)^2}}$$

Substitution

$$\boxed{\begin{aligned} u &= 2x \\ du &= 2\,dx \\ \tfrac{1}{2}\,du &= dx \end{aligned}}$$

$$= \tfrac{1}{2}\int \frac{du}{\sqrt{1^2-u^2}}$$

$$= \tfrac{1}{2}\sin^{-1} u = \tfrac{1}{2}\sin^{-1} 2x$$

$$\therefore \int_0^{1/4} \frac{dx}{\sqrt{1-4x^2}} = \tfrac{1}{2}\Big[\sin^{-1} 2x\Big]_0^{1/4}$$

$$= \tfrac{1}{2}\Big[\sin^{-1}\tfrac{1}{2} - \sin^{-1} 0\Big]$$

$$= \frac{1}{2}\left[\frac{\pi}{6} - 0\right] = \frac{\pi}{12}$$

(ii) $$\int \frac{dx}{4+9x^2} = \int \frac{dx}{2^2+(3x)^2}$$

Substitution

$$\boxed{\begin{aligned} u &= 3x \\ du &= 3\,dx \\ \tfrac{1}{3}\,du &= dx \end{aligned}}$$

$$= \tfrac{1}{3}\int \frac{du}{2^2+u^2}$$

$$= \tfrac{1}{3}\cdot\tfrac{1}{2}\tan^{-1}\frac{u}{2} = \tfrac{1}{6}\tan^{-1}\frac{3x}{2}$$

$$\therefore \int_0^{2/3} \frac{dx}{4+9x^2} = \tfrac{1}{6}\left[\tan^{-1}\frac{3x}{2}\right]_0^{2/3}$$

$$= \tfrac{1}{6}\Big[\tan^{-1} 1 - \tan^{-1} 0\Big]$$

$$= \frac{1}{6}\left[\frac{\pi}{4} - 0\right] = \frac{\pi}{24}$$

Exercise 14.7

Find:

1. $\displaystyle\int \frac{dx}{\sqrt{4-x^2}}$
2. $\displaystyle\int \frac{dx}{\sqrt{9-x^2}}$
3. $\displaystyle\int \frac{dx}{\sqrt{25-x^2}}$
4. $\displaystyle\int \frac{dx}{16+x^2}$
5. $\displaystyle\int \frac{3\,dx}{9+x^2}$
6. $\displaystyle\int \frac{10\,dx}{25+x^2}$

In questions 7–12, use the suggested substitution:

7. $\displaystyle\int \frac{dx}{9+4x^2}$, $u = 2x$
8. $\displaystyle\int \frac{dx}{16+9x^2}$, $u = 3x$
9. $\displaystyle\int \frac{dx}{\sqrt{4-25x^2}}$, $u = 5x$
10. $\displaystyle\int \frac{dx}{1+16x^2}$, $u = 4x$
11. $\displaystyle\int \frac{2\,dx}{1+4x^2}$, $u = 2x$
12. $\displaystyle\int \frac{4\,dx}{\sqrt{9-16x^2}}$, $u = 4x$

Evaluate each of the following:

13. $\displaystyle\int_0^2 \frac{dx}{4+x^2}$
14. $\displaystyle\int_0^3 \frac{dx}{9+x^2}$
15. $\displaystyle\int_0^1 \frac{dx}{\sqrt{4-x^2}}$
16. $\displaystyle\int_2^4 \frac{dx}{\sqrt{16-x^2}}$
17. $\displaystyle\int_{1/\sqrt{3}}^1 \frac{dx}{1+x^2}$
18. $\displaystyle\int_{1/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}}$
19. $\displaystyle\int_0^{1/3} \frac{3\,dx}{\sqrt{1-9x^2}}$
20. $\displaystyle\int_0^{2/3} \frac{dx}{4+9x^2}$
21. $\displaystyle\int_0^{2/5} \frac{dx}{\sqrt{4-25x^2}}$
22. If $\displaystyle\int_0^k \frac{dx}{x^2+25} = \frac{\pi}{20}$, find the value of k, $k \in \mathbf{R}$.

23. If $\int_{3/2}^{a} \frac{dx}{\sqrt{9-x^2}} = \frac{\pi}{3}$, find the value of a, $a \in \mathbf{R}$.

24. If $\int_{0}^{p} \frac{dx}{16+9x^2} = \frac{\pi}{48}$, find the value of p, $p \in \mathbf{R}$.

More difficult variations on the above standard integrals involve completing the square.

Example

(i) Express $x^2 - 4x + 29$ in the form $(x-p)^2 + q^2$ and, hence, evaluate $\int_2^7 \frac{dx}{x^2 - 4x + 29}$.

(ii) Express $3 - 2x - x^2$ in the form $a^2 - (x+b)^2$ and, hence, evaluate $\int_{-1}^{0} \frac{dx}{\sqrt{3-2x-x^2}}$.

Solution:

(i)

$$x^2 - 4x + 29$$
$$= x^2 - 4x + 4 + 25$$
$$= (x-2)^2 + 5^2$$
$$\int \frac{dx}{x^2 - 4x + 29} = \int \frac{dx}{(x-2)^2 + 5^2}$$

Substitution

$$\boxed{\begin{aligned} u &= x - 2 \\ du &= dx \end{aligned}}$$

$$= \int \frac{du}{5^2 + u^2} = \frac{1}{5}\tan^{-1}\frac{u}{5} = \frac{1}{5}\tan^{-1}\left(\frac{x-2}{5}\right)$$
$$\therefore \int_2^7 \frac{dx}{x^2 - 4x + 29} = \frac{1}{5}\left[\tan^{-1}\left(\frac{x-2}{5}\right)\right]_2^7$$
$$= \tfrac{1}{5}\left[\tan^{-1} 1 - \tan^{-1} 0\right]$$
$$= \frac{1}{5}\left[\frac{\pi}{4} - 0\right] = \frac{\pi}{20}$$

(ii)

$$3 - 2x - x^2 \qquad \text{(add 1 and subtract 1)}$$
$$= 3 - (x^2 + 2x)$$
$$= 3 + 1 - (x^2 + 2x + 1)$$
$$= 4 - (x+1)^2 = 2^2 - (x+1)^2$$
$$\int \frac{dx}{\sqrt{3 - 2x - x^2}} = \int \frac{dx}{\sqrt{2^2 - (x+1)^2}}$$

Substitution

$$\boxed{\begin{aligned} u &= x + 1 \\ du &= dx \end{aligned}}$$

$$= \int \frac{dx}{\sqrt{2^2 - u^2}} = \sin^{-1}\frac{u}{2} = \sin^{-1}\left(\frac{x+1}{2}\right)$$
$$\therefore \int_{-1}^{0} \frac{dx}{\sqrt{3 - 2x - x^2}} = \left[\sin^{-1}\left(\frac{x+1}{2}\right)\right]_{-1}^{0}$$
$$= \sin^{-1}\tfrac{1}{2} - \sin^{-1} 0$$
$$= \frac{\pi}{6} - 0 = \frac{\pi}{6}$$

Exercise 14.8

Express each of the following in the form $(x+p)^2 + q^2$:

1. $x^2 + 4x + 13$ **2.** $x^2 + 8x + 25$ **3.** $x^2 - 6x + 10$ **4.** $x^2 - 2x + 17$

Express each of the following in the form $a^2 - (x+b)^2$:

5. $8 - 2x - x^2$ **6.** $7 - 6x - x^2$ **7.** $11 - 10x - x^2$ **8.** $8x - x^2$

Evaluate each of the following:

9. $\displaystyle\int_{-1}^{3} \frac{dx}{16+(x+1)^2}$ **10.** $\displaystyle\int_{-2}^{1} \frac{dx}{x^2-2x+10}$ **11.** $\displaystyle\int_{1}^{4} \frac{dx}{\sqrt{9-(x-1)^2}}$

12. $\displaystyle\int_{0}^{1} \frac{dx}{\sqrt{3-2x-x^2}}$ **13.** $\displaystyle\int_{-1}^{0} \frac{4\,dx}{x^2+2x+2}$ **14.** $\displaystyle\int_{-2}^{1} \frac{2\,dx}{\sqrt{5-4x-x^2}}$

15. Find the values of p and q such that $x^2-4x+13=(x-p)^2+q^2$.

Hence, evaluate $\displaystyle\int_{2}^{3} \frac{dx}{x^2-4x+13}$, giving your answer in the form $k\tan^{-1}k$.

16. Write $(1-x)(7+x)$ in the form $a^2-(x+b)^2$ and, hence, evaluate $\displaystyle\int_{-5}^{1} \frac{3\,dx}{\sqrt{(1-x)(7+x)}}$.

17. Find the real number k given that $\displaystyle\int_{k}^{3} \frac{dx}{x^2-2x+5}=\frac{\pi}{8}$.

Integrals of the Form $\int\sqrt{a^2-x^2}\,dx$

Integrals of the form $\int\sqrt{a^2-x^2}\,dx$ can be calculated by the substitution:

$$\boxed{x=a\sin\theta}$$

This substitution removes the square root and turns the integral into a trigonometrical substitution.

Note: It is simpler to change the limits to the corresponding limits for θ.
Integrals of this form are not in the tables.

Example

Evaluate: $\displaystyle\int_{0}^{4}\sqrt{16-x^2}\,dx$.

Substitution

$$x=4\sin\theta$$
$$dx=4\cos\theta\,d\theta$$

Limits

$x=4$	$x=0$
$4\sin\theta=4$	$4\sin\theta=0$
$\sin\theta=1$	$\sin\theta=0$
$\theta=\frac{\pi}{2}$	$\theta=0$

Solution:

$$\int\sqrt{16-x^2}\,dx=\int\sqrt{4^2-x^2}\,dx$$

$$16-x^2=16-16\sin^2\theta=16(1-\sin^2\theta)=16\cos^2\theta$$

$$\therefore\quad \sqrt{16-x^2}=\sqrt{16\cos^2\theta}=4\cos\theta$$

$$\therefore\quad \int_0^4\sqrt{16-x^2}\,dx$$

$$=\int_0^{\pi/2} 4\cos\theta\,.\,4\cos\theta\,d\theta$$

$$=16\int_0^{\pi/2}\cos^2\theta\,d\theta=16\int_0^{\pi/2}\tfrac{1}{2}(1+\cos 2\theta)\,d\theta=8\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta$$

$$=8\left[\theta+\tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2}$$

$$=8\left[\left(\frac{\pi}{2}+0\right)-(0-0)\right]=8\left(\frac{\pi}{2}\right)=4\pi$$

Exercise 14.9

Evaluate each of the following, in each case using the suggested substitution:

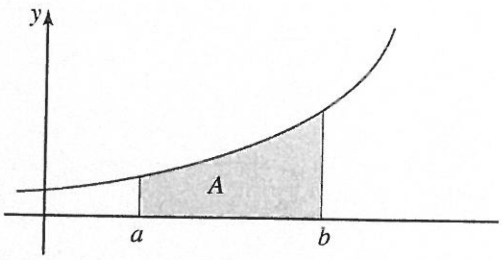
1. $\int_0^2 \sqrt{4-x^2}\, dx, \quad x = 2\sin\theta$
2. $\int_0^3 \sqrt{9-x^2}\, dx, \quad x = 3\sin\theta$
3. $\int_0^5 \sqrt{25-x^2}\, dx, \quad x = 5\sin\theta$
4. $\int_0^{1/\sqrt{2}} \sqrt{1-x^2}\, dx, \quad x = \sin\theta$
5. $\int_0^{1/2} \sqrt{1-4x^2}\, dx, \quad 2x = \sin\theta$
6. $\int_0^{2/3} \sqrt{4-9x^2}\, dx, \quad 3x = 2\sin\theta$
7. $\int_0^{3/4} \sqrt{9-16x^2}\, dx, \quad 4x = 3\sin\theta$
8. $\int_0^{\sqrt{2}} \sqrt{4-x^2}\, dx, \quad x = 2\sin\theta$
9. $\int_0^3 \sqrt{36-x^2}\, dx, \quad x = 6\sin\theta$
10. $\int_0^{2\sqrt{3}} \sqrt{16-x^2}\, dx, \quad x = 4\sin\theta$
11. Verify that $\int_0^{3/2} \sqrt{3-x^2}\, dx = \frac{\pi}{2} + \frac{3\sqrt{3}}{8}$. (Hint: Let $x = \sqrt{3}\sin\theta$).

Area Under a Curve

Area between a curve and the x-axis:

The area, A, of the region bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is given by:

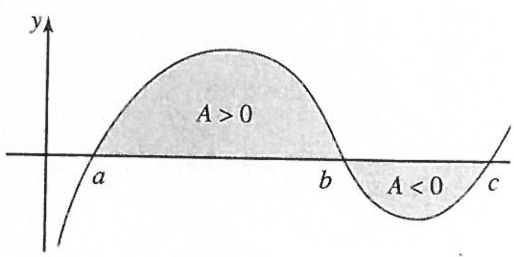
$$A = \int_a^b y\, dx \quad \text{or} \quad A = \int_a^b f(x)\, dx$$



This is positive if the area is above the x-axis.
This is negative if the area is below the x-axis.

If the curve cuts the x-axis between the limits, then:

(i) find the areas above and below the x-axis separately;
(ii) add these two values together.

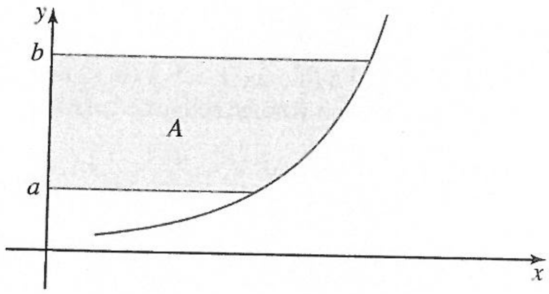


Area between a curve and the y-axis:

The area, A, bounded by the curve $y = f(x)$, the y-axis and the lines $y = a$ and $y = b$ is given by:

$$A = \int_a^b x \, dy$$

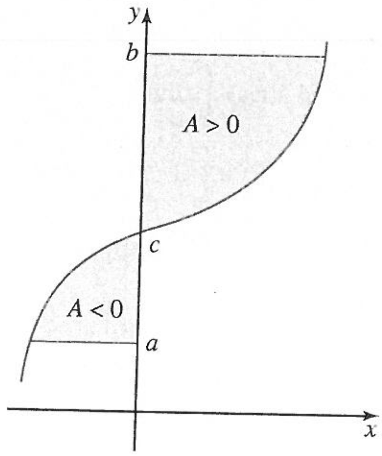
In this case, x must be expressed as a function of y before we can integrate.



This is positive if the area is to the right of the y-axis.
This is negative if the area is to the left of the y-axis.

If the curve cuts the y-axis between the limits, then:

(i) find the areas to the right and the left of the y-axis separately;
(ii) add these two values together.



If not given, it is good practice to draw a sketch of the function and check to see if the curve cuts the x-axis, or y-axis, between the given limits. If the curve is completely above, or below, the x-axis between the limits, we can evaluate the integral between the limits given. If the curve is completely to the right, or left, of the y-axis between the limits, we can evaluate the integral between the limits given.

Example

$C : y = 3x - x^2$ and $L : y = x$ represent a curve and a line respectively. Find the coordinates of the points where the curve and line intersect and draw a rough sketch of C and L.
Find the area bounded by the curve C, the line L and the x-axis.

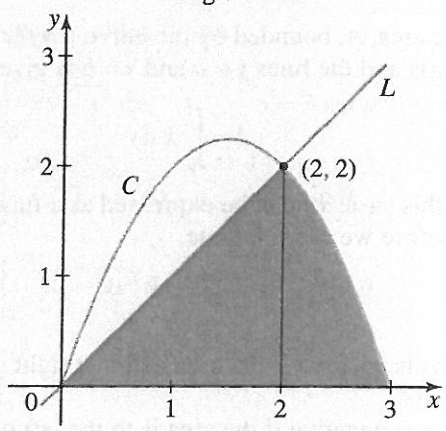
Solution:

$$\begin{aligned} \text{curve} &= \text{line} \\ 3x - x^2 &= x \\ -x^2 + 2x &= 0 \\ x^2 - 2x &= 0 \\ x(x-2) &= 0 \end{aligned}$$

$x = 0 \quad$ or $\quad x = 2$

As $y = x$,
when $x = 0$, $y = 0$ and when $x = 2$, $y = 2$.
Thus, the curve and line intersect at (0, 0) and (2, 2).
We also need to find where the curve cuts the x-axis.

Rough sketch



(Shaded region is the required area)

On the x-axis: $\quad y = 0$

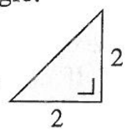
$$\therefore \quad 3x - x^2 = 0$$
$$x^2 - 3x = 0$$
$$x(x-3) = 0$$
$$x = 0 \quad \text{or} \quad x = 3$$

Thus the curve cuts the x-axis at $x = 0$ and $x = 3$.

$$\text{Shaded area} = \int_0^2 (\text{line})\, dx + \int_2^3 (\text{curve})\, dx$$
$$= \int_0^2 x\, dx + \int_2^3 (3x - x^2)\, dx$$
$$= \left[\frac{x^2}{2}\right]_0^2 + \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_2^3$$
$$= \left[(\tfrac{4}{2}) - (0)\right] + \left[(\tfrac{27}{2} - \tfrac{27}{3}) - (\tfrac{12}{2} - \tfrac{8}{3})\right]$$
$$= 2 + \tfrac{7}{6} = \tfrac{19}{6}$$

Note: The area bounded by the line and the x-axis between $x = 0$ and $x = 2$ could also have been ca culated by calculating the area of a triangle.

$$\int_0^2 (\text{line})\, dx = \int_0^2 x\, dx =$$



$$= \tfrac{1}{2}(2)(2) = 2.$$

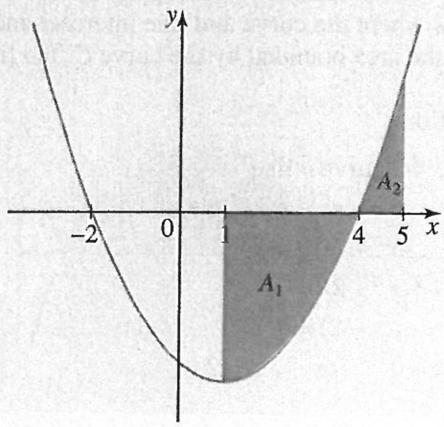
Example

Find the area bounded by the curve $y = x^2 - 2x - 8$, the x-axis and the lines $x = 1$ to $x = 5$.

Solution:

First make a sketch of the curve $y = x^2 - 2x - 8$.
It cuts the x-axis at $y = 0$.

Thus, $\quad x^2 - 2x - 8 = 0$
$$(x-4)(x+2) = 0$$
$$x = 4 \quad \text{or} \quad x = -2$$



The graph is shown on the right. The sketch shows that the required area is in two parts. One part lies below the x-axis, A_1, and is negative, the other part lies above the x-axis, A_2, and is positive. Thus, we calculate the two areas separately.

$$A_1 = \int_1^4 (x^2 - 2x - 8)\,dx$$
$$= \left[\frac{x^3}{3} - x^2 - 8x\right]_1^4$$
$$= \left[\tfrac{64}{3} - 16 - 32\right] - \left[\tfrac{1}{3} - 1 - 8\right]$$
$$= -\tfrac{80}{3} + \tfrac{26}{3} = -18 = 18$$
(as area must be positive)

$$A_2 = \int_4^5 (x^2 - 2x - 8)\,dx$$
$$= \left[\frac{x^3}{3} - x^2 - 8x\right]_4^5$$
$$= \left[\tfrac{125}{3} - 25 - 40\right] - \left[\tfrac{64}{3} - 16 - 32\right]$$
$$= -\tfrac{70}{3} + \tfrac{80}{3} = \tfrac{10}{3}$$

Thus, Area $= A_1 + A_2 = 18 + \tfrac{10}{3} = \tfrac{64}{3}$

Area between two curves:

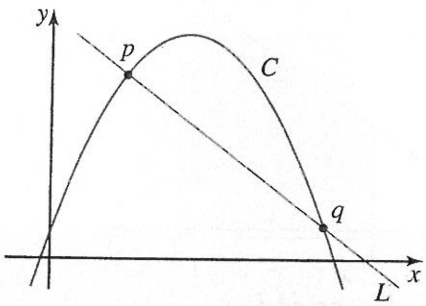
1. We need to find where the curves intersect.
2. Subtract the areas under the curves between the points of intersection.

This can be done by the evaluation of one integral using the x coordinates of the point of intersection as the limits.

Example

The diagram shows part of the curve $C: y = 3 + 6x - x^2$ and the line $L: y = 15 - 2x$

(i) Find the x coordinates of p and q.
(ii) Calculate the area bounded by C and L.

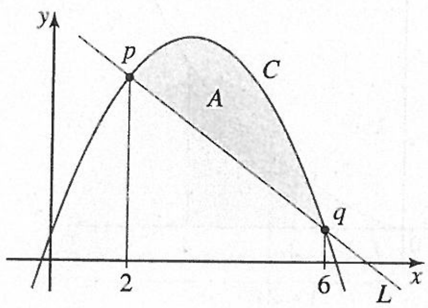


Solution:

(i) We need to find the x coordinates where the curve and line intersect.

$C: y = 3 + 6x - x^2 \qquad L: y = 15 - 2x$

Thus,
$$3 + 6x - x^2 = 15 - 2x$$
$$-x^2 + 8x - 12 = 0$$
$$x^2 - 8x + 12 = 0$$
$$(x-2)(x-6) = 0$$
$$x = 2 \quad \text{or} \quad x = 6$$



(ii) The shaded region represents the area bounded by the curve, C, and the line, L, i.e. the are under the curve ***less*** the area under the line, between the limits $x = 2$ and $x = 6$.

$$\text{Shaded area} = A = \int_2^6 (3 + 6x - x^2)\,dx - \int_2^6 (15 - 2x)\,dx$$

$$= \int_2^6 (3 + 6x - x^2 - 15 + 2x)\,dx$$

$$= \int_2^6 (8x - x^2 - 12)\,dx$$

$$= \left[4x^2 - \frac{x^3}{3} - 12x\right]_2^6$$

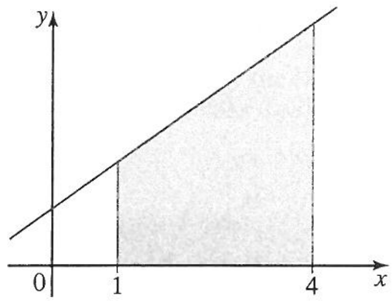
$$= (144 - 72 - 72) - (16 - \tfrac{8}{3} - 24)$$

$$= 10\tfrac{2}{3}$$

Exercise 14.10

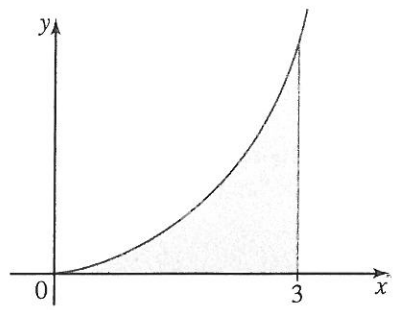
Calculate the area of the shaded region in each of the following diagrams:

1.



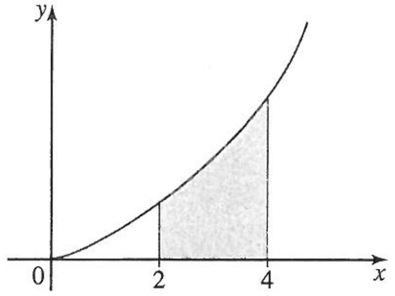
$y = x + 1$

2.



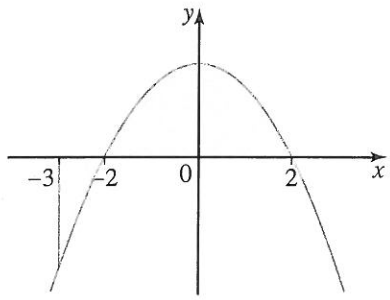
$y = x^2$

3.



$y = 2x^3$

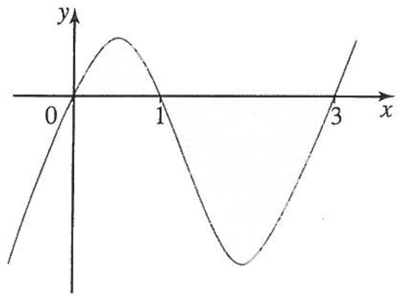
4.



$y = 4 - x^2$

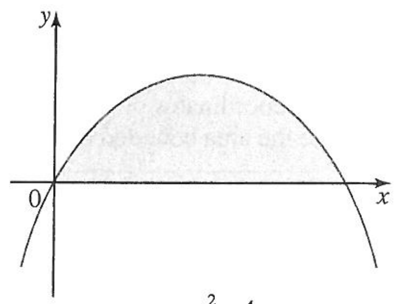
. the area

5.



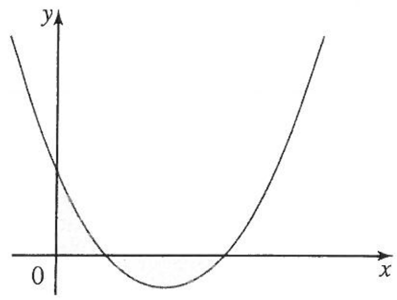
$y = x^3 - 4x^2 + 3x$

6.



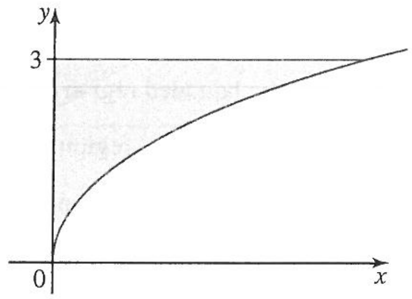
$y = x^2 - 4x$

7.



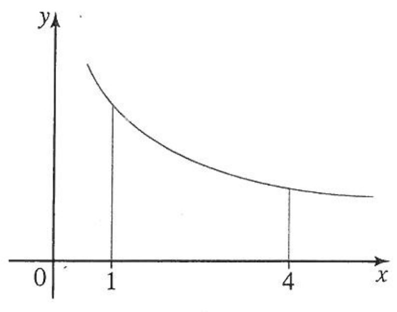
$y = x^2 - 5x + 4$

8.



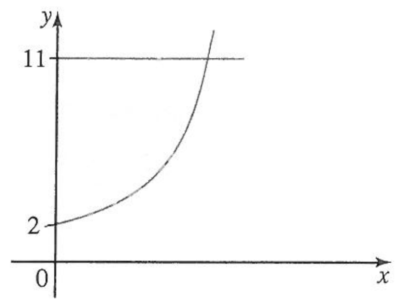
$y = \sqrt{x}$

9.



$y = \dfrac{1}{x^2} + 3$

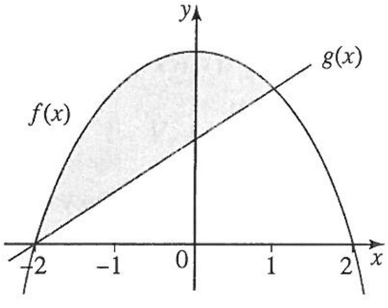
10.



$y = x^2 + 2$

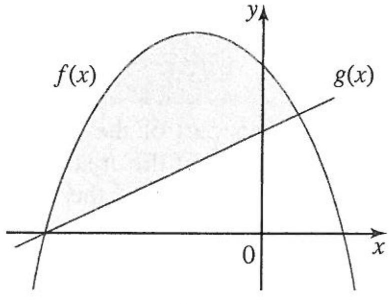
In questions 11 and 12, $f(x)$ is the curve and $g(x)$ is the line:

11.



$f(x) = 4 - x^2$ and $g(x) = x + 2$

12.

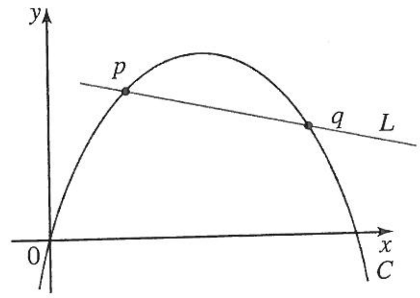


$f(x) = 6 - x - x^2$ and $g(x) = x + 3$

13. The diagram shows part of the curve $C: y = 5x - x^2$ and the line $L: y = 8 - x$

(i) Find the x coordinates of p and q.

(ii) Calculate the area bounded by C and L.



14. $f(x) = x^2 + 3$ and $g(x) = 3x + 1$. $f(x)$ and $g(x)$ meet at the points a and b.

(i) Find the coordinates of a and b.

(ii) Sketch $f(x)$ and $g(x)$ on the same axes and scales.

(iii) Find the area of the bounded region enclosed by $f(x)$ and $g(x)$.

15. Find the area of the bounded region enclosed by the curve $y = 4x - x^2$ and the lines $x = 0$ and $x = 5$

16. Find the area of the bounded region enclosed by the curve $y = 5x - 2x^2$ and the line $y = x$.

17. Find the area of the bounded region enclosed by the curve $y = x^2 - 3x + 3$ and the line $y = 2x - 1$.

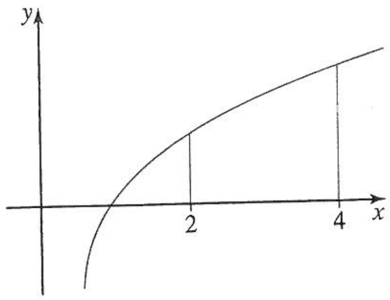
18. Find the area enclosed by the curve $y = x^2 + 1$ and the line $y = 5$.

19. Find the area enclosed by the curve $y = x^2$ and the y-axis from $y = 1$ to $y = 4$.

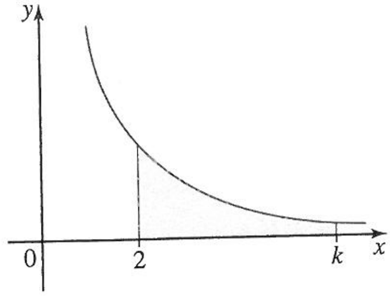
20. The diagram shows part of the graph of the function $f(x) = \dfrac{x^2 - 1}{x}$.

Calculate the area of the shaded region.

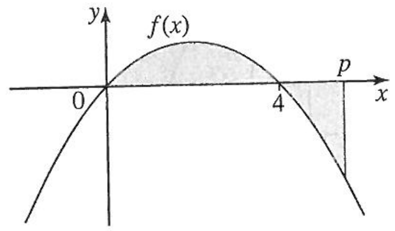
Write your answer in the form $a - \ln b$.



21. The diagram shows part of the graph of the curve $f(x) = \dfrac{12}{x^2}$. If the region bounded by $f(x)$, the x-axis and lines $x = 2$ and $x = k$, $k > 2$, is 4, find the value of k.



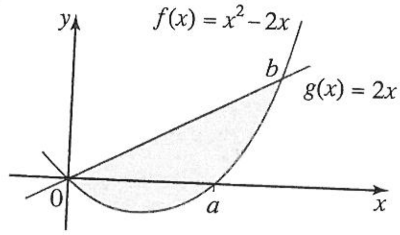
22. The diagram shows part of the graph of the function $f(x) = 4x - x^2$. If the areas of the two shaded regions are equal, find the value of p, $p > 0$.



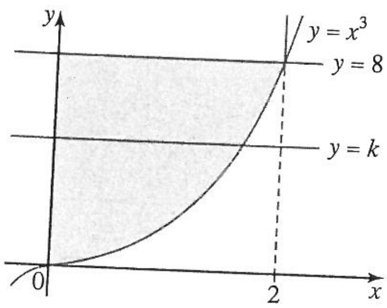
23. The diagram shows a sketch of the functions:

$f(x) = x^2 - 2x$ and $g(x) = 2x$.
$f(x)$ cuts the x-axis at the points o and a.
$f(x)$ and $g(x)$ meet at the point b.

(i) Find the coordinates of a and b.
(ii) Find the area of the shaded region.



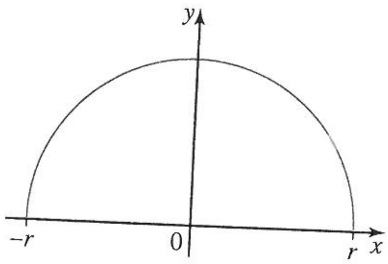
24. **(i)** If $y = x^3$, express x as a function of y.
(ii) The region bounded by the y-axis, the curve $y = x^3$ and the line $y = 8$ is divided into two regions of equal area by the line $y = k$.
Show that $k^4 = 512$



25. **(i)** Evaluate $\int_0^6 \sqrt{36 - x^2}\, dx$ (let $x = 6 \sin \theta$)

(ii) $x^2 + y^2 = r^2$ is the equation of a circle, centre the origin and radius r.
Express y in terms of r and x.

(iii) The diagram shows the graph of the function $y = \sqrt{r^2 - x^2}$.
Using integration methods, prove that the area of circle of radius r is πr^2.



Volumes of Revolution

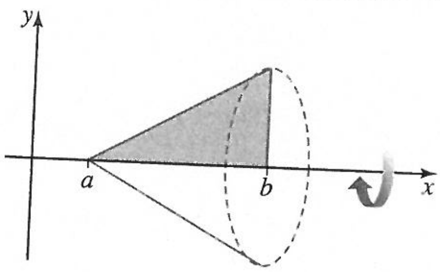
If an area is rotated about the x- or y-axis, the three-dimensional object formed is called a '**solid of revolution**' and its volume is a '**volume of revolution**'.

Rotation about the x-axis:

The volume, V_x, generated by rotating, once, the area bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is given by:

$$V_x = \pi \int_a^b y^2\, dx$$

A triangle would generate a cone.

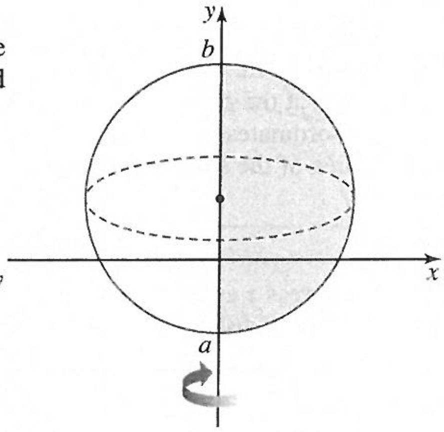


Rotation about the y-axis:

The volume, V_y, generated by rotating, once, the area bounded by the curve $y = f(x)$, the y-axis and the lines $y = a$ and $y = b$ is given by:

$$V_y = \pi \int_a^b x^2 \, dy$$

A semi-circle would generate a sphere.
In this case, x must be expressed as a function of y first.



On our course, the solids of revolution are confined to cones, or parts of cones, and spheres, or p[arts] of spheres.

Example

Find the volume of the solid generated by rotating about the y-axis the area bounded by the y-axi[s], the line $2x - y + 2 = 0$ and the lines $y = 4$ and $y = 7$.

Solution:

The diagram on the right represents the situation.
The line $2x - y + 2 = 0$ cuts the x-axis at $(-1, 0)$ and the y-axis at $(2, 0)$.

$$V_y = \pi \int_a^b x^2 \, dy$$

$$= \pi \int_4^7 \tfrac{1}{4}(y^2 - 4y + 4) \, dy$$

$$= \frac{\pi}{4} \int_4^7 (y^2 - 4y + 4) \, dy$$

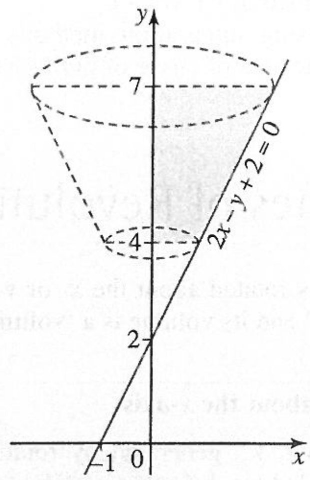
$$= \frac{\pi}{4} \left[\frac{y^3}{3} - 2y^2 + 4y \right]_4^7$$

$$= \frac{\pi}{4} \left[\left(\tfrac{343}{3} - 98 + 28\right) - \left(\tfrac{64}{3} - 32 + 16\right) \right]$$

$$= \frac{\pi}{4} \left[\tfrac{133}{3} - \tfrac{16}{3} \right]$$

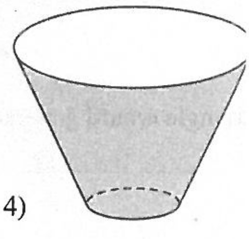
$$= \frac{\pi}{4} [39]$$

$$= \tfrac{39}{4}\pi$$



Generates a frustrum

$$2x - y + 2 = 0$$
$$2x = y - 2$$
$$x = \tfrac{1}{2}(y - 2)$$
$$x^2 = \tfrac{1}{4}(y - 2)^2$$
$$x^2 = \tfrac{1}{4}(y^2 - 4y + 4)$$



Example

Using integration methods, prove that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

Solution:

$$V_x = \pi\int_a^b y^2\,dx$$

$$= \pi\int_{-r}^{r} (r^2 - x^2)\,dx$$

$$= \pi\left[r^2x - \frac{x^3}{3}\right]_{-r}^{r}$$

$$= \pi\left[\left(r^3 - \frac{r^3}{3}\right) - \left(-r^3 + \frac{r^3}{3}\right)\right]$$

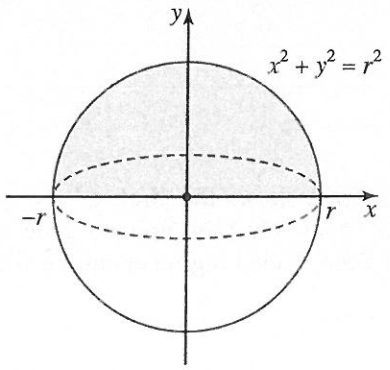
$$= \pi\left(r^3 - \frac{r^3}{3} + r^3 - \frac{r^3}{3}\right)$$

$$= \pi(\tfrac{4}{3}r)$$

$$= \tfrac{4}{3}\pi r^3$$

Note: r is a constant.

Consider the circle $x^2 + y^2 = r^2$.
We rotate the top half (semi-circle) of the circle about the x-axis. This generates a sphere.

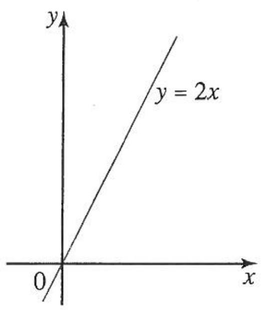


The circle cuts the x-axis at $-r$ and r.
Thus, the limits of integration are $-r$ and r.

$$x^2 + y^2 = r^2$$
$$y^2 = (r^2 - x^2)$$

Exercise 14.11

1. The diagram shows the line $y = 2x$.
Find the volume of the cone generated by rotating about the x-axis the area bounded by the line $y = 2x$ and the lines $x = 0$ and $x = 4$.



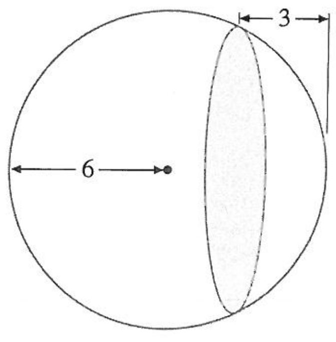
2. Sketch the line $x + y - 2 = 0$.
Find the volume of the solid generated by rotating about the y-axis the area bounded by the y-axis, the line $x + y - 2 = 0$ and the lines $x = 0$ and $x = 2$.

3. Find the volume of the solid generated by rotating about the y-axis the area bounded by the y-axis, the line $2x - y - 2 = 0$ and the lines $y = 2$ and $y = 6$.

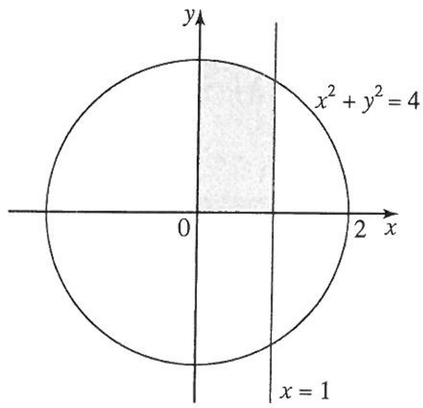
4. The equation of the line K is $3x + 2y - 6 = 0$. Using integration methods, find the volume of the cones generated by rotating the areas bounded by K and the axes about:
(i) the x-axis **(ii)** the y-axis.

5. Using integration methods, find the volume generated by rotating about the x-axis the circle $x^2 + y^2 = 9$.

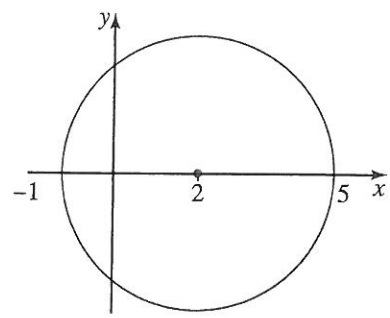
6. A sphere has a radius of 6 cm.
A spherical cap of depth 3 cm, is removed from the sphere, as shown.
Using integration methods, find the volume of the spherical cap.



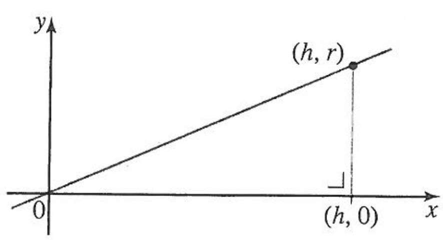
7. The diagram shows the circle $x^2 + y^2 = 4$ and the line $x = 1$. Find the volume generated by rotating the shaded region about the x-axis.



8. The diagram shows a graph of the circle $(x-2)^2 + y^2 = 9$.
If $y^2 = a + bx - x^2$, find the value of a and b.
Find the volume of the solid generated by rotating about the x-axis the area bounded by the x-axis, the circle $(x-2)^2 + y^2 = 9$ and the lines $x = 3$ and $x = 4$.



9. A triangle has vertices $(0, 0)$, (h, r), $(h, 0)$, as shown.
Find the equation of the line which contains the points $(0, 0)$ and (h, r). Hence, using integration methods, prove that the volume of a right-circular cone of base radius r and height h is $\frac{1}{3}\pi r^2 h$.



10. Using integration methods, prove that the volume of a hemisphere of radius r is $\frac{2}{3}\pi r^3$.

11. Using integration methods, find the volume of water needed to fill a hemispherical bowl of radius 9 cm to a depth of 6 cm.

x-axis the circle

owl of radius

12. The diagram shows the circle $x^2 + y^2 = 4$ and the line $x = 1$. The line intercepts the circle at the points a and b.

(i) Find the coordinates of a and b.

(ii) Find the volume generated by rotating the shaded sector about the y-axis.