CHAPTER 12

DIFFERENTIAL CALCULUS 2
DIFFERENTIATION BY RULE

Differentiation by Rule

Differentiation from first principles can become tedious and difficult. Fortunately, it is not always necessary to use first principles. There are a few rules (which can be derived from first principles) which enable us to write down the derivative of a function quite easily.

Rule 1: General Rule

If:

$$y = x^n \quad \text{then} \quad \frac{dy}{dx} = nx^{n-1}$$
$$y = ax^n \quad \text{then} \quad \frac{dy}{dx} = nax^{n-1}$$

In words:

Multiply by the power and reduce the power by 1.

Example

Differentiate with respect to x:

(i) $y = x^5$ **(ii)** $y = -3x^2$ **(iii)** $y = 5x$ **(iv)** $y = \frac{8}{x^2}$

(v) $y = 6\sqrt{x}$ **(vi)** $y = \frac{2}{\sqrt{x}}$ **(vii)** $y = \frac{6}{x^{1/3}}$ **(viii)** $y = 7$

Solution:

(i) $y = x^5$ $\quad \frac{dy}{dx} = 5x^{5-1} = 5x^4$

(ii) $y = -3x^2$ $\quad \frac{dy}{dx} = 2 \times -3x^{2-1} = -6x$

(iii) $y = 5x = 5x^1$ $\quad \frac{dy}{dx} = 1 \times 5x^{1-1} = 5x^0 = 5 \quad (x^0 = 1)$

(iv) $y = \frac{8}{x^2} = 8x^{-2}$ $\quad \frac{dy}{dx} = -2 \times 8x^{-2-1} = -16x^{-3} = -\frac{16}{x^3}$

(v) $y = 6\sqrt{x} = 6x^{1/2}$ $\quad \frac{dy}{dx} = \frac{1}{2} \times 6x^{1/2-1} = 3x^{-1/2} = \frac{3}{x^{1/2}} = \frac{3}{\sqrt{x}}$

(vi) $y = \frac{2}{\sqrt{x}} = 2x^{-1/2}$ $\quad \frac{dy}{dx} = -\frac{1}{2} \times 2x^{-1/2-1} = -1x^{-3/2} = -\frac{1}{x^{3/2}}$

(vii) $y = \frac{6}{x^{1/3}} = 6x^{-1/3}$ $\quad \frac{dy}{dx} = -\frac{1}{3} \times 6x^{-1/3-1} = -2x^{-4/3} = -\frac{2}{x^{4/3}}$

(viii) $y = 7 = 7x^0$ $\quad \frac{dy}{dx} = 0 \times 7x^{0-1} = 0$

Part (viii) leads to the rule:

The derivative of a constant = 0.

Note: The line $y = 7$ is a horizontal line. Its slope is 0.
Therefore its derivative (also its slope) equals 0.
In other words, the derivative of a constant always equals zero.

Sum or Difference

If the expression to be differentiated contains more than one term, just differentiate, separately, eac term in the expression.

Example

Find $f'(x)$ for each of the following:

(i) $f(x) = x + \frac{1}{x^2}$

(ii) $f(x) = \frac{2}{\sqrt{x}} - \frac{1}{x^4} + 5$

Solution:

(i) $f(x) = x + \frac{1}{x^2}$

$f(x) = x + x^{-2}$

$f'(x) = 1 - 2x^{-3}$

$f'(x) = 1 - \frac{2}{x^3}$

(ii) $f(x) = \frac{2}{\sqrt{x}} - \frac{1}{x^4} + 5$

$f(x) = 2x^{-1/2} - x^{-4} + 5$

$f'(x) = -x^{-3/2} + 4x^{-5}$

$f'(x) = -\frac{1}{x^{3/2}} + \frac{4}{x^5}$

parately, each

Evaluating Derivatives

Often we have to evaluate a derivative for a particular value.

Example

(i) If $s = 3t^2 + 5t - 7$, find the value of $\dfrac{ds}{dt}$ when $t = 2$.

(ii) If $f(x) = \sqrt{x} + 3x$, evaluate $f'(4)$.

Solution:

(i)

$$s = 3t^2 + 5t - 7$$

$$\frac{ds}{dt} = 6t + 5$$

$$\left.\frac{ds}{dt}\right|_{t=2} = 6(2) + 5$$

$$= 12 + 5 = 17$$

(ii)

$$f(x) = \sqrt{x} + 3x$$

$$f(x) = x^{1/2} + 3x$$

$$f'(x) = \tfrac{1}{2}x^{-1/2} + 3$$

$$= \frac{1}{2\sqrt{x}} + 3$$

$$f'(4) = \frac{1}{2\sqrt{4}} + 3$$

$$= \frac{1}{4} + 3 = 3\tfrac{1}{4}$$

$\dfrac{ds}{dt}$ is the derivative of s with respect to t. $\dfrac{dA}{dr}$ is the derivative of A with respect to r.

Second Derivatives

The derivative of $\dfrac{dy}{dx}$, that is $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$, is denoted by $\dfrac{d^2y}{dx^2}$ and is called the

'second derivative of y with respect to x'.

$\dfrac{d^2y}{dx^2}$ is pronounced 'dee two y, dee x squared'.

The derivative of $f'(x)$ is denoted by $f''(x)$ and is called the

'second derivative of $f(x)$ with respect to x'.

Example

(i) If $f(x) = x + \dfrac{1}{x}$, find $f''(x)$ and $f''(2)$.

(ii) If $h = 10 + 30t^2 - 4t^3$, evaluate $\dfrac{d^2h}{dt^2}$ when $t = 3$.

Solution:

(i)

$$f(x) = x + \frac{1}{x}$$
$$f(x) = x + x^{-1}$$
$$f'(x) = 1 - x^{-2}$$
$$f''(x) = 2x^{-3}$$
$$= \frac{2}{x^3}$$
$$\therefore\ f''(2) = \frac{2}{2^3} = \frac{2}{8} = \frac{1}{4}$$

(ii)

$$h = 10 + 30t^2 - 4t^3$$
$$\frac{dh}{dt} = 60t - 12t^2$$
$$\frac{d^2h}{dt^2} = 60 - 24t$$
$$\left.\frac{d^2h}{dt^2}\right|_{t=3} = 60 - 24(3) = -12$$

Example

If $y = x^4$, show that $\dfrac{4y}{3}\left(\dfrac{d^2y}{dx^2}\right) - \left(\dfrac{dy}{dx}\right)^2 = 0$.

Solution:

$$y = x^4$$
$$\frac{dy}{dx} = 4x^3$$
$$\frac{d^2y}{dx^2} = 12x^2$$

$$\frac{4y}{3}\left(\frac{d^2y}{dx^2}\right) - \left(\frac{dy}{dx}\right)^2$$
$$= \frac{4x^4}{3}(12x^2) - (4x^3)^2$$
$$= 16x^6 - 16x^6$$
$$= 0$$

Note: $\left(\dfrac{dy}{dx}\right)^2 \neq \dfrac{d^2y}{dx^2}$

Exercise 12.1

Differentiate each of the following with respect to x:

1. x^3
2. $3x^4$
3. $-5x^2$
4. $3x$
5. $-2x$
6. 5
7. -3
8. $\dfrac{1}{x^2}$
9. $\dfrac{2}{x^3}$
10. $-\dfrac{2}{x^5}$

11. $6x^{1/3}$ **12.** $\frac{1}{x}$ **13.** $\sqrt{x}$ **14.** $\frac{4}{\sqrt{x}}$ **15.** $\frac{1}{x^{2/3}}$

16. $x^3 - 5x$ **17.** $1 - x^2$ **18.** $x^2 - \frac{5}{x}$ **19.** $2x^2 - \frac{3}{x^4}$ **20.** $\frac{1}{x^2} + \frac{1}{x}$

21. $x^4 - \frac{2}{x^2}$ **22.** $6\sqrt{x} - \frac{2}{\sqrt{x}}$ **23.** $\frac{3}{x} + \frac{2}{x^2} + \frac{6}{x^{1/3}}$ **24.** $\frac{2}{x} - \frac{1}{\sqrt{x}} + \frac{3}{x^{1/3}}$

Find $\frac{d^2y}{dx^2}$ for each of the following:

25. $y = 4x^3 + 6x^2$ **26.** $y = x^2 - x^4$ **27.** $y = 6x^3 - 12x^2 - 8x + 4$

28. $y = \frac{1}{x}$ **29.** $y = x^2 - \frac{8}{x}$ **30.** $y = \sqrt{x}$

31. $y = \frac{1}{\sqrt{x}} + \sqrt{x}$ **32.** $y = 8\sqrt{x} - \frac{1}{x^2}$ **33.** $y = 9x^{1/3} + \frac{18}{x^{1/3}}$

34. If $f(x) = 3x^2 - 4x - 7$, evaluate **(i)** $f'(2)$ **(ii)** $f''(-1)$.

35. If $f(x) = -4\sqrt{x}$, evaluate $f''(9)$.

36. If $A = 3r^2 - 5r$, find the value of $\frac{dA}{dr}$ when $r = 3$.

37. If $s = 3t - 2t^2$, find the value of **(i)** $\frac{ds}{dt}$ **(ii)** $\frac{d^2s}{dt^2}$ when $t = 2$.

38. If $V = 3h - h^2 - 3h^3$, find $\frac{dV}{dh}$ when $h = 1$.

39. If $A = \pi r^2$, find $\frac{dA}{dr}$ when $\frac{r}{5} = 1$.

40. If $V = \frac{4}{3}\pi r^3$, find $\frac{dV}{dr}$ when $2r - 5 = 0$.

41. $f(x) = 3x^2 - 4x$. If $f'(k) = 8$, find the value of k, $k \in \mathbf{R}$.

42. $f(x) = x^3 + 1$. If $f''(a) = 18$, find the value of a, $a \in \mathbf{R}$.

43. If $y = 3x^2 + 2x$, show that $y\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - 6x = 0$.

44. If $y = 4x^3 - 6x^2$, show that $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 12x = 0$.

Find the values of x for which **(i)** $\frac{dy}{dx} = 0$ **(ii)** $\frac{d^2y}{dx^2} = 0$.

45. If $y = \frac{1}{x^2}$, show that $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 - 10y^3 = 0$.

Product, Quotient and Chain Rules

Rule 2: Product Rule

Suppose u and v are functions of x.

If $y = uv$,

then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$.

In words:

First by the derivative of the second + second by the derivative of the first.

Example

If $y = (x^2 - 3x + 2)(x^2 - 2)$, find $\dfrac{dy}{dx}$.

Solution:

Let $u = x^2 - 3x + 2$ and let $v = x^2 - 2$

$\dfrac{du}{dx} = 2x - 3$ and $\dfrac{dv}{dx} = 2x$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \qquad \text{(product rule)}$$

$$= (x^2 - 3x + 2)(2x) + (x^2 - 2)(2x - 3)$$

$$= 2x^3 - 6x^2 + 4x + 2x^3 - 3x^2 - 4x + 6$$

$$= 4x^3 - 9x^2 + 6$$

Rule 3: Quotient Rule

Suppose u and v are functions of x.

If $y = \dfrac{u}{v}$

then $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$.

In words:

$$\frac{\text{Bottom by the derivative of the top} - \text{Top by the derivative of the bottom}}{(\text{Bottom})^2}$$

Example

If $y=\dfrac{x^2}{x-2}$, find $\dfrac{dy}{dx}$.

Solution:

$$\text{Let } u = x^2 \qquad \text{and} \qquad \text{let} \qquad v = x-2$$

$$\frac{du}{dx} = 2x \qquad \text{and} \qquad \frac{dv}{dx} = 1$$

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} \qquad \text{(quotient rule)}$$

$$= \frac{(x-2)(2x)-(x^2)(1)}{(x-2)^2}$$

$$= \frac{2x^2-4x-x^2}{(x-2)^2}$$

$$= \frac{x^2-4x}{(x-2)^2}$$

Note: It is usual practice to simplify the top but **not** the bottom.

Function of a function

When we write, for example, $y = (x+5)^3$, we say that y is a function of x.
If we let $u = (x+5)$, then $y = u^3$, where $u = (x+5)$.
We say that y is a function u, and u is a function of x.
The new variable, u, is the **link** between the two expressions.

Rule 4: Chain Rule

Suppose u is a function of x.

If $\quad y = u^n$

then $\quad \dfrac{dy}{dx} = nu^{n-1}\dfrac{du}{dx}$.

The chain rule should be done in **one** step.

Example

Find $\frac{dy}{dx}$ for each of the following:

(i) $y = (x^2 - 3x)^4$

(ii) $y = \frac{3}{2x+5}$

(iii) $y = \sqrt{4x-3}$

(iv) $y = \left(x^2 + \frac{1}{x}\right)^3$

Solution:

(i)

$$y = (x^2 - 3x)^4$$

$$\frac{dy}{dx} = 4(x^2 - 3x)^3(2x - 3)$$

$$= (8x - 12)(x^2 - 3x)^3$$

(ii)

$$y = \frac{3}{2x+5}$$

$$y = 3(2x+5)^{-1}$$

$$\frac{dy}{dx} = -3(2x+5)^{-2}(2)$$

$$= \frac{-6}{(2x+5)^2}$$

(iii)

$$y = \sqrt{4x-3}$$

$$y = (4x-3)^{1/2}$$

$$\frac{dy}{dx} = \tfrac{1}{2}(4x-3)^{-1/2}(4)$$

$$= \frac{2}{(4x-3)^{1/2}} = \frac{2}{\sqrt{4x-3}}$$

(iv)

$$y = \left(x^2 + \frac{1}{x}\right)^3$$

$$y = (x^2 + x^{-1})^3$$

$$\frac{dy}{dx} = 3(x^2 + x^{-1})^2(2x - x^{-2})$$

$$= 3\left(x^2 + \frac{1}{x}\right)^2\left(2x - \frac{1}{x^2}\right)$$

Often we have to deal with a combination of the product, quotient or chain rules.

Example

Find $\frac{dy}{dx}$ if **(i)** $y = x\sqrt{9-x^2}$ **(ii)** $y = \sqrt{\frac{1-x}{1+x}}$

Solution:

(i)

$$y = x\sqrt{9-x^2}$$

$$y = x(9-x^2)^{1/2}$$

$$\frac{dy}{dx} = (x)\,.\,\underbrace{\tfrac{1}{2}(9-x^2)^{-1/2}(-2x)}_{\text{(chain rule here)}} + (9-x^2)^{1/2}(1) \qquad \text{(product rule and chain rule)}$$

$$= -x^2(9-x^2)^{-1/2} + (9-x^2)^{1/2}$$

$$= \frac{-x^2}{\sqrt{9-x^2}} + \sqrt{9-x^2}$$

(ii) $y = \sqrt{\dfrac{1-x}{1+x}}$

$$y = \left(\frac{1-x}{1+x}\right)^{1/2}$$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{-1/2}\left[\frac{(1+x)(1)-(1-x)(1)}{(1+x)^2}\right] \qquad \left(\begin{array}{c}\text{chain rule followed by}\\ \text{the quotient rule}\end{array}\right)$$

$$= \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{1/2}\left[\frac{-1-x-1+x}{(1+x)^2}\right] \qquad \left(\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}\right)$$

$$= \frac{(1+x)^{1/2}}{2(1-x)^{1/2}} \cdot \frac{-2}{(1+x)^2}$$

$$= \frac{-1}{(1-x)^{1/2}(1+x)^{3/2}}$$

Exercise 12.2

In questions 1 to 6, use the product rule to find $\dfrac{dy}{dx}$:

1. $y = (2x+3)(x-4)$ **2.** $y = (x+5)(x^2-3x+2)$ **3.** $y = (3x-4)(x^2-2x+3)$

4. $y = (x+3)(x^2-6x+8)$ **5.** $y = (5x^2-3x)(x^2-5x)$ **6.** $y = (3x^3-2x^2+4)(2x-1)$

In questions 7 to 12, use the quotient rule to find $\dfrac{dy}{dx}$:

7. $y = \dfrac{3x+2}{x+1}$ **8.** $y = \dfrac{2x-1}{x+3}$ **9.** $y = \dfrac{3x-1}{x^2-2}$

10. $y = \dfrac{x^2-1}{x^2+1}$ **11.** $y = \dfrac{1-x}{2x-x^2}$ **12.** $y = \dfrac{x^2-x-6}{x^2+x-6}$

In questions 13–18, use the chain rule to find $\dfrac{dy}{dx}$:

13. $y = (3x+2)^4$ **14.** $y = (x^2+2x)^3$ **15.** $y = (2x^2+1)^5$

16. $y = \sqrt{4x+2}$ **17.** $y = \dfrac{1}{2x-5}$ **18.** $y = \dfrac{1}{\sqrt{2x^2-4x}}$

Find $\dfrac{dy}{dx}$ if:

19. $y = x^2(x+3)^4$ **20.** $y = 3x(x+2)^3$ **21.** $y = 3x^2(2x+3)^2$

22. $y = x^2\sqrt{2x+1}$ **23.** $y = x\sqrt{1+x^2}$ **24.** $y = \sqrt{\dfrac{x+1}{x}}$

25. If $f(x) = \sqrt{\dfrac{x}{x+3}}$, find the value of $f'(1)$. **26.** If $f(x) = \sqrt{\dfrac{x-1}{x+1}}$, find the value of $f'(\frac{5}{4})$.

Differentiation of Trigonometric Functions

The rules for differentiating also apply to trigonometric functions.
The following are in the tables on page 41, but they are shown only for x.
The chain rule is used throughout, assuming u is a function of x.
Therefore, if you are using the tables, replace x with u and **always** multiply by $\frac{du}{dx}$.

Basic rule (page 41 tables)	
$f(x)$	$f'(x)$
$\cos x$	$-\sin x$
$\sin x$	$\cos x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\cot x$	$-\operatorname{cosec}^2 x$

Chain rule	
$f(u)$	$f'(u) \cdot \frac{du}{dx}$
$\cos u$	$-\sin u \cdot \frac{du}{dx}$
$\sin u$	$\cos u \cdot \frac{du}{dx}$
$\tan u$	$\sec^2 u \cdot \frac{du}{dx}$
$\sec u$	$\sec u \tan u \cdot \frac{du}{dx}$
$\operatorname{cosec} u$	$-\operatorname{cosec} u \cot u \cdot \frac{du}{dx}$
$\cot u$	$-\operatorname{cosec}^2 u \cdot \frac{du}{dx}$

Example

Find the derivatives of the functions:

(i) $\cos 3x$ **(ii)** $\tan^3 5x$ **(iii)** $x \sin x$ **(iv)** $\sqrt{\cos x}$

Solution:

(i)
$$y = \cos 3x$$
$$\frac{dy}{dx} = (-\sin 3x)(3)$$
$$= -3 \sin 3x$$

(ii)
$$t = \tan^3 5x$$
$$y = (\tan 5x)^3$$
$$\frac{dy}{dx} = 3(\tan 5x)^2(\sec^2 5x)(5)$$
[PTA: (power) (trig. function) (angle)]
$$= 15 \tan^2 5x \sec^2 5x$$

(iii)
$$y = x \sin x$$
(use the product rule)
$$\frac{dy}{dx} = (x)(\cos x) + (\sin x)(1)$$
$$= x \cos x + \sin x$$

(iv)
$$y = \sqrt{\cos x}$$
$$y = (\cos x)^{1/2}$$
(chain rule)
$$\frac{dy}{dx} = \tfrac{1}{2}(\cos x)^{-1/2}(-\sin x)$$
$$= \frac{-\sin x}{2\sqrt{\cos x}}$$

Example

If $f(x) = \dfrac{x^2}{x + \cos x}$, evaluate $f'\left(\dfrac{\pi}{2}\right)$.

Solution:

$$f(x) = \frac{x^2}{x + \cos x}$$

$$f'(x) = \frac{(x + \cos x)(2x) - x^2(1 - \sin x)}{(x + \cos x)^2} \qquad \text{(quotient rule)}$$

$$f'\left(\frac{\pi}{2}\right) = \frac{\left(\frac{\pi}{2} + \cos\frac{\pi}{2}\right)(\pi) - \left(\frac{\pi}{2}\right)^2\left(1 - \sin\frac{\pi}{2}\right)}{\left(\frac{\pi}{2} + \cos\frac{\pi}{2}\right)^2} \qquad \left(\begin{array}{l}\text{Don't simplify:}\\ \text{put in } x = \frac{\pi}{2}\end{array}\right)$$

$$= \frac{\left(\frac{\pi}{2} + 0\right)(\pi) - \left(\frac{\pi^2}{4}\right)(1 - 1)}{\left(\frac{\pi}{2} + 0\right)^2} \qquad \left(\cos\frac{\pi}{2} = 0,\ \sin\frac{\pi}{2} = 1\right)$$

$$= \frac{\left(\frac{\pi}{2}\right)(\pi) - \left(\frac{\pi^2}{4}\right)(0)}{\left(\frac{\pi}{2}\right)^2} = \frac{\frac{\pi^2}{2}}{\frac{\pi^2}{4}} = \frac{2\pi^2}{\pi^2} = 2$$

Exercise 12.3

Find $\dfrac{dy}{dx}$ if:

1. $y = \sin 4x$
2. $y = \cos 3x$
3. $y = \tan 2x$
4. $y = \sec 5x$
5. $y = -\operatorname{cosec} 6x$
6. $y = -2 \cot 4x$
7. $y = \sin(2x - 3)$
8. $y = \tan(3x + 2)$
9. $y = 2 \tan x + \sec x$
10. $y = x^2 \sin x$
11. $y = 3x \tan x$
12. $y = x^2 \cos 2x$
13. $y = \dfrac{\sin x}{x}$
14. $y = \dfrac{1}{1 - \sin x}$
15. $y = \dfrac{1 + \sin x}{\cos x}$
16. $y = \cos^3 x$
17. $y = \sin^2 4x$
18. $y = \tan^4 3x$
19. $y = (1 + \sin^2 x)^3$
20. $y = \sqrt{\sin x}$
21. $y = \sqrt{\cos 2x}$
22. $f(x) = \dfrac{\cos x + \sin x}{\cos x - \sin x}$. Show that $f'(x) = \dfrac{2}{1 - \sin 2x}$.

23. If $y = \cos 3x$, show that $\dfrac{d^2y}{dx^2} = -9y$.

24. If $y = 3\cos x + \sin x$, show that:

(i) $\cos x\left(\dfrac{dy}{dx}\right) + y\sin x - 1 = 0$ **(ii)** $\dfrac{d^2y}{dx^2} - 3\left(\dfrac{dy}{dx}\right) + 2y - 10\sin x = 0$.

25. If $f(x) = \sin x \cos x$, evaluate $f'\left(\dfrac{\pi}{4}\right)$.

26. If $y = \cos 2x + 2\sin x$, evaluate $\dfrac{dy}{dx}$ at $x = \dfrac{\pi}{6}$.

27. $f(x) = \dfrac{\sin x}{1 + \tan x}$. Evaluate $f'(0)$.

Implicit Differentiation

If $y = f(x)$, the variable y is given **explicitly** (clearly) in terms of x.
For example, $y = x^3 - 2x^2 + 5x - 4$ is an explicit function.
Some curves are defined by implicit functions, that is, functions which cannot be expressed in the forn $y = f(x)$.
For example, $x^2 + xy + y^3 = 7$ is an **implicit function.**
It cannot be written in the form $y = f(x)$.
It is for this reason that we must have a method for differentiating explicit functions.
An implicit function involving x and y can be differentiated with respect to x as it stands, using th chain rule.

Method for differentiating implicit functions:

1. Differentiate, term by term, on both sides with respect to x.

2. Bring all terms with $\dfrac{dy}{dx}$ to the left and bring all other terms to the right.

3. Make $\dfrac{dy}{dx}$ the subject of the equation.

It is useful to remember that, by the chain rule,

$$\frac{d}{dx}(y^2) = 2y\frac{dy}{dx} \quad \text{and} \quad \frac{d}{dx}(y^3) = 3y^2\frac{dy}{dx}$$

as y is considered as a function of x.

$$\frac{d}{dx}(y^n) = ny^{n-1}\left(\frac{dy}{dx}\right)$$

ed in the form

ds, using the

Example

Given that $2x^3 + 3xy^2 - y^3 + 6 = 0$, evaluate $\frac{dy}{dx}$ at the point $(-1, 1)$.

Solution:

(use product rule here)

$$2x^3 + 3xy^2 - y^3 + 6 = 0$$

$$6x^2 + 3\left[x \,.\, 2y\frac{dy}{dx} + y^2(1)\right] - 3y^2\frac{dy}{dx} = 0$$

$$6x^2 + 6xy\frac{dy}{dx} + 3y^2 - 3y^2\frac{dy}{dx} = 0$$

(divide each term by -3)

$$6xy\frac{dy}{dx} - 3y^2\frac{dy}{dx} = -6x^2 - 3y^2$$

$$\frac{dy}{dx}(6xy - 3y^2) = -6x^2 - 3y^2$$

$$\frac{dy}{dx} = \frac{-6x^2 - 3y^2}{6xy - 3y^2} = \frac{2x^2 + y^2}{y^2 - 2xy} = \frac{2x^2 + y^2}{y(y - 2x)}$$

$$\left.\frac{dy}{dx}\right|_{\substack{x=-1\\y=1}} = \frac{2(-1)^2 + (1)^2}{1(1+2)} = \frac{3}{3} = 1$$

Note: To evaluate $\frac{dy}{dx}$ we used both coordinates of the point.

Exercise 12.4

For each of the following curves, express $\frac{dy}{dx}$ in terms of x and y:

1. $x^2 + y^2 = 4$ **2.** $x^2 + 2y - y^2 = 5$ **3.** $x^2 - 6y^3 + y = 0$

4. $x^2 + y^2 - 4x - 6y + 9 = 0$ **5.** $x^2 + xy + y^2 = 13$ **6.** $x^2 + 3xy + 2y^2 = 6$

7. $x^2y - 5x = 2$ **8.** $xy^2 + x^2 = 2$ **9.** $x^2y + xy^2 = 2$

Find the value of $\frac{dy}{dx}$ at the point specified:

10. $x^2 + y^2 = 25$ at the point $(3, -4)$ **11.** $x^2 + xy + 2y^2 = 28$ at the point $(2, -4)$

12. $x^2 + 4xy - 2y^2 - 8 = 0$ at the point $(0, 2)$ **13.** $x^3 + y^2 + 3x^2y = 21$ at the point $(2, 1)$

14. Find the slope of the tangent to the curve $y^2 + 3xy + 2x^2 = 6$ at the point $(1, 1)$.

15. Find the slope of the tangent to the curve $x \sin y + y^2 = 1 + \frac{\pi^2}{4}$ at the point $\left(1, \frac{\pi}{2}\right)$.

Note: $\frac{d}{dx}(\sin y) = \cos y \frac{dy}{dx}$.

Parametric Differentiation

If x and y are each expressed in terms of a third variable, t say (or θ), called the **parameter**, then $x = f(t)$ and $y = g(t)$ give the parametric forms of the equation relating to x and y respectively.

To find $\frac{dy}{dx}$ do the following:

1. Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$, separately. **2.** Use $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

Example

(i) If $x = 4t^3$ and $y = (1 + 3t^2)^2$, express $\frac{dy}{dx}$ in terms of t.

Hence, or otherwise, evaluate $\frac{dy}{dx}$ when $t = -1$.

(ii) Let $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$; show that $\frac{dy}{dx} = \tan\theta$.

$$\left[a \neq 0, -\pi < \theta < \pi \qquad \text{and} \qquad \theta \neq \pm\frac{\pi}{2}\right]$$

Solution:

$x = 4t^3$

$$\frac{dx}{dt} = 12t^2$$

$y = (1 + 3t^2)^2$

(chain rule)

$$\frac{dy}{dt} = 2(1 + 3t^2)^1(6t) = 12t(1 + 3t^2)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{12t(1 + 3t^2)}{12t^2} = \frac{1 + 3t^2}{t}$$

$$\left.\frac{dy}{dx}\right|_{t=-1} = \frac{1 + 3(-1)^2}{-1} = \frac{1 + 3}{-1} = \frac{4}{-1} = -4$$

(ii) $x = a(\cos\theta + \theta\sin\theta)$

$$\frac{dx}{d\theta} = a(-\sin\theta + \theta.\cos\theta + \sin\theta.1)$$
$$= a(-\sin\theta + \theta\cos\theta + \sin\theta)$$
$$= a\theta\cos\theta$$

$y = a(\sin\theta - \theta\cos\theta)$

$$\frac{dy}{d\theta} = a[\cos\theta - (\theta.-\sin\theta + \cos\theta.1)]$$
$$= a(\cos\theta + \theta\sin\theta - \cos\theta)$$
$$= a\theta\sin\theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

Exercise 12.5

Find $\frac{dy}{dx}$, in terms of t, if:

1. $x = 2t,\quad y = t^2$
2. $x = 2t + 3,\quad y = 2t^3$
3. $x = t^2 + 1,\quad y = t^3$
4. $x = 3t^4,\quad y = 2t^2 + 5$
5. $x = t(1 - t),\quad y = t(1 - t^2)$
6. $x = 6t + 5,\quad y = (2t - 1)^3$
7. $x = \frac{1}{t},\quad y = t^2 + 4t$
8. $x = 2\sqrt{t},\quad y = 5t + 4$
9. $x = 1 + \frac{1}{t},\quad y = t + \frac{1}{t}$
10. $x = \frac{t^2}{1 + t^3},\quad y = \frac{t^3}{1 + t^3}$
11. $x = \frac{t-2}{t+1}$ and $y = \frac{t+2}{t+1}$. If $\frac{dy}{dx} = k$, find the value of k.
12. If $x = \frac{2}{t}$ and $y = 3t^2 - 1$, express $\frac{dy}{dx}$ in terms of t. Evaluate $\frac{dy}{dx}$ at the point (2, 2).
13. If $x = \frac{3t-1}{t}$ and $y = \frac{t^2+4}{t}$, express $\frac{dy}{dx}$ in terms of t.

    Find the values of t for which $\frac{dy}{dx} = 0$.
14. If $x = 2t + \sin 2t$ and $y = \cos 2t$, show that $\frac{dy}{dx} = -\tan t$.
15. If $x = \sec\theta$ and $y = \tan\theta$, show that $\frac{dy}{dx} = \operatorname{cosec}\theta$.
16. If $x = k(\theta - \sin\theta)$ and $y = k(1 - \cos\theta)$, $k \in \mathbf{R}$, find $\frac{dy}{dx}$.
17. Given $y = \sin\theta\cos\theta - \theta$ and $x = 2\cos\theta$, show that **(i)** $\frac{dy}{d\theta} = -2\sin^2\theta$ **(ii)** $\frac{dy}{dx} = \sin\theta$.
18. If $x = 3\cos\theta - 4\sin\theta$ and $y = 4\cos\theta + 3\sin\theta$, evaluate $\frac{dy}{dx}$ at $\theta = \frac{\pi}{2}$.

19. If $x = \sin\theta$ and $y = \sin n\theta$, where $n \in \mathbf{R}$, show that $(1 - x^2)\left(\frac{dy}{dx}\right)^2 - n^2(1 - y^2) = 0$.

20. $x = k(1 + \cos\theta)$, $\quad y = 2k\sin^2\theta$, where $0 \leqslant \theta \leqslant \pi$ and k is a positive constant.

(i) Find $\frac{dy}{dx}$ in the form $p\cos\theta$ where $p \in \mathbf{Z}$.

(ii) Find, in terms of k, the coordinates of the point q where $\theta = \tan^{-1}\frac{\sqrt{7}}{3}$.

Differentiation of Inverse Trigonometric Functions

The rules for differentiating also apply to inverse trigonometric functions.
The following are in the tables on page 41, but they are shown only for x.
The chain rule is used throughout, assuming u is a function of x.

Replace a with 1, x with u, and always multiply by $\frac{du}{dx}$.

Basic rule (page 41 tables)	
$f(x)$	$f'(x)$
$\sin^{-1}\frac{x}{a}$	$\frac{1}{\sqrt{a^2 - x^2}}$
$\tan^{-1}\frac{x}{a}$	$\frac{a}{a^2 + x^2}$

Chain rule	
$f(u)$	$f'(u)\cdot\frac{du}{dx}$
$\sin^{-1} u$	$\frac{1}{\sqrt{1 - u^2}}\cdot\frac{du}{dx}$
$\tan^{-1} u$	$\frac{1}{1 + u^2}\cdot\frac{du}{dx}$

Note: The derivative of $\cos^{-1} u$ is **not** in the syllabus.

Example

(i) If $y = \tan^{-1}\left(\frac{x}{1+x}\right)$, show that $\frac{dy}{dx} = \frac{1}{2x^2 + 2x + 1}$, $\quad x \neq -1$.

(ii) Given $y = \sin^{-1}(3x - 1)$, calculate the value of $\frac{dy}{dx}$ at $x = \frac{1}{3}$.

Solution:

$$y = \tan^{-1}\left(\frac{x}{1+x}\right)$$

(quotient rule)

$$\frac{dy}{dx} = \frac{1}{1 + \left(\frac{x}{1+x}\right)^2}\cdot\left(\frac{(1+x)(-1) - (x)(1)}{(1+x)^2}\right)$$

$$\boxed{\begin{gathered} y = \tan^{-1} u \\ \frac{dy}{dx} = \frac{1}{1+u^2}\cdot\frac{du}{dx} \end{gathered}}$$

$$= \frac{1}{1 + \frac{x^2}{(1+x)^2}} \cdot \left(\frac{1+x-x}{(1+x)^2}\right)$$

$$= \frac{(1+x)^2}{(1+x)^2 + x^2} \cdot \frac{1}{(1+x)^2} \qquad \left(\begin{array}{l}\text{multiply the top and bottom} \\ \text{of the first fraction by } (1+x)^2\end{array}\right)$$

$$= \frac{1}{(1+x)^2 + x^2}$$

$$= \frac{1}{1 + 2x + x^2 + x^2}$$

$$= \frac{1}{2x^2 + 2x + 1}$$

(ii)

$$y = \sin^{-1}(3x-1)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(3x-1)^2}} \cdot (3)$$

$$= \frac{3}{\sqrt{1-(3x-1)^2}}$$

$$\boxed{\begin{array}{c} y = \sin^{-1} u \\ \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-u^2}} \cdot \dfrac{du}{dx} \end{array}}$$

$$\left.\frac{dy}{dx}\right|_{x=\frac{1}{3}} = \frac{3}{\sqrt{1-(0)^2}} = \frac{3}{\sqrt{1}} + \frac{3}{1} = 3$$

Exercise 12.6

Find $\frac{dy}{dx}$ for each of the following:

1. $y = \sin^{-1} 2x$

2. $y = \tan^{-1} 3x$

3. $y = \sin^{-1}(x-1)$

4. $y = \tan^{-1}(2x+1)$

5. $y = \tan^{-1} x^2$

6. $y = \sin^{-1} 2x^3$

7. $y = (\sin^{-1} 5x)^2$

8. $y = \tan^{-1}\left(\frac{x}{3}\right)$

9. $y = \sin^{-1}\left(\frac{x}{2}\right)$

10. $y = \sin^{-1}(\cos x)$

11. $y = x \sin^{-1} x$

12. $y = 6x \tan^{-1} 2x$

13. Given $y = \sin^{-1}(4x-1)$, calculate the value of $\frac{dy}{dx}$ at $x = \frac{1}{4}$.

14. Given $y = \tan^{-1}\left(\frac{1}{x}\right)$, show that $\frac{dy}{dx} = -\frac{1}{1+x^2}$.

15. Given $y = \tan^{-1}(\cos x)$, calculate the value of $\frac{dy}{dx}$ at $x = \frac{\pi}{6}$.

16. If $y = \tan^{-1}\left(\frac{x}{a}\right)$, show that $\frac{dy}{dx} = \frac{a}{a^2+x^2}$.

17. Explain why $p\sqrt{1-q} = \sqrt{p^2 - p^2 q}$, $\quad p, q \in \mathbf{R}$.

If $y = \sin^{-1}\left(\frac{x}{a}\right)$, show that $\frac{dy}{dx} = \frac{1}{\sqrt{a^2-x^2}}$.

18. $f(x) = \dfrac{1}{x}\sin^{-1}\left(\dfrac{1}{x}\right)$. Show that $f'(\sqrt{2}) = -\dfrac{1}{2} - \dfrac{\pi}{8}$.

19. If $y = \tan^{-1} x$, show that $\dfrac{d^2y}{dx^2}(1 + x^2) + 2x\dfrac{dy}{dx} = 0$.

20. If $u = \dfrac{1+x}{1-x}$, show that $\dfrac{du}{dx} = \dfrac{2}{(1-x)^2}$.

Hence, if $y = \tan^{-1}\left(\dfrac{1+x}{1-x}\right)$, find $\dfrac{dy}{dx}$.

Verify that $\quad 2x\left(\dfrac{dy}{dx}\right)^2 + \dfrac{d^2y}{dx^2} = 0$.

21. Explain why $\sqrt{a} = \dfrac{a}{\sqrt{a}}$, $a \in \mathbf{R}$, $a \neq 0$.

Given $y = \sin^{-1} x + x\sqrt{1 - x^2}$, show that $\quad \dfrac{dy}{dx} = 2\sqrt{1 - x^2}$.

Differentiation of Exponential Functions

The rules for differentiating apply also to exponential functions.
Exponent is another word for index. A function such as $y = 2^x$, in which the variable occurs as an index, is called 'an exponential function'.
The function $y = e^x$ is called '**the exponential function**' or '**natural exponential function**'.
e is an irrational constant whose value is 2.71828 correct to six significant figures.
e^x is the only basic function which is its own derivative. That is:

$$\boxed{\text{If } y = e^x, \quad \frac{dy}{dx} = e^x.}$$

Note: The positive number e behaves just like other positive numbers such as 2 or 5.
e^x obeys all the usual laws of indices or exponents.

Using the chain rule:

Suppose u is a function of x.

If $\quad y = e^u$

then $\quad \dfrac{dy}{dx} = e^u \cdot \dfrac{du}{dx}$.

Example

Find $\dfrac{dy}{dx}$ if **(i)** $y = e^{x^2-3x}$ **(ii)** $y = \dfrac{2}{e^{3x}}$ **(iii)** $y = e^{\sin 2x}$ **(iv)** $y = \dfrac{x}{e^{2x}}$

Solution:

(i)

$$y = e^{x^2-3x}$$
$$\frac{dy}{dx} = e^{x^2-3x}(2x-3)$$
$$= (2x-3)e^{x^2-3x}$$

(ii)

$$y = \frac{2}{e^{3x}} = 2e^{-3x}$$
$$\frac{dy}{dx} = 2e^{-3x}(-3)$$
$$= -6e^{-3x} = -\frac{6}{e^{3x}}$$

(iii)

$$y = e^{\sin 2x}$$
$$\frac{dy}{dx} = e^{\sin 2x}(\cos 2x)(2)$$
$$= (2\cos 2x)e^{\sin 2x}$$

(iv)

$$y = \frac{x}{e^{2x}} = xe^{-2x}$$

(use the product rule)

$$\frac{dy}{dx} = xe^{-2x}(-2) + e^{-2x}(1)$$
$$= -2xe^{-2x} + e^{-2x}$$
$$= e^{-2x}(1-2x) = \frac{1-2x}{e^{2x}}$$

Note: The quotient rule could also be used.

Example

If $y = xe^{-x}$, show that $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 0$.

Solution:

$$y = xe^{-x}$$
$$\frac{dy}{dx} = x[e^{-x}(-1)] + e^{-x}(1) \qquad \text{(product rule)}$$
$$= -xe^{-x} + e^{-x}$$
$$= e^{-x}(1-x)$$

$$\frac{d^2y}{dx^2} = e^{-x}(-1) + (1-x)[e^{-x}(-1)] \qquad \text{(product rule, again)}$$
$$= -e^{-x} + (1-x)(-e^{-x})$$
$$= -e^{-x} - e^{-x} + xe^{-x}$$
$$= xe^{-x} - 2e^{-x}$$
$$= e^{-x}(x-2)$$

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y$$
$$= e^{-x}(x-2) + 2[e^{-x}(1-x)] + (xe^{-x})$$
$$= xe^{-x} - 2e^{-x} + 2e^{-x} - 2xe^{-x} + xe^{-x}$$
$$= e^{-x}(x-2+2-2x+x) = e^{-x}(0) = 0.$$

Exercise 12.7

Find $\dfrac{dy}{dx}$ for each of the following:

1. $y = e^{4x}$
2. $y = 2e^{3x}$
3. $y = e^{x^2}$
4. $y = e^{x^2-5x}$
5. $y = e^{4x^2}$
6. $y = e^{-x}$
7. $y = \dfrac{5}{e^{2x}}$
8. $y = \dfrac{2}{e^{x^2}}$
9. $y = e^{\sin x}$
10. $y = e^{\cos 2x}$
11. $y = e^{4 \tan x}$
12. $y = e^{x \sin x}$
13. $y = xe^{x}$
14. $y = x^2 e^{5x}$
15. $y = e^{2x} \cos x$
16. $y = e^{-x^2} \sin x$
17. $y = \dfrac{x^2}{e^{2x}}$
18. $y = (3 + e^{x^2})^4$
19. $y = \dfrac{1}{3 - e^{2x^2}}$
20. $y = \sqrt{1 - 2e^{4x}}$
21. If $f(x) = \dfrac{1+e^x}{1-e^x}$, show that $f'(x) = \dfrac{2e^x}{(1-e^x)^2}$.
22. If $f(\theta) = e^{1 + \sin\theta}$, evaluate **(i)** $f'(0)$ **(ii)** $f''($
23. If $y = e^{2x}$, show that $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 0$.
24. If $y = xe^{-2x}$, show that $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = 0$.
25. If $y = e^x \sin x$, show that $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 0$.
26. Given that $y = x + \sin^{-1} x$, show that $(1 - x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + x = 0$.
27. If $y = e^{kx}$, find the values of $k \in \mathbf{R}$ for which $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 0$.
28. If $y = e^{2t}$ and $x = e^t$, show that $\dfrac{dy}{dx} = 2e^t$.
29. If $y = te^t$ and $x = t^2e^t$, show that $\dfrac{dy}{dx} = \dfrac{t+1}{t(t+2)}$.
30. Given $y = e^\theta \cos\theta$ and $x = e^\theta \sin\theta$, where $-\dfrac{3\pi}{4} < \theta < \dfrac{\pi}{4}$, show that $\left(\dfrac{dy}{d\theta}\right)^2 + \left(\dfrac{dx}{d\theta}\right)^2 = 2e^{2\theta}$.

    Evaluate $\dfrac{dy}{dx}$ at $\theta = \dfrac{\pi}{2}$.
31. If $y = e^{-nx} \cos kx$, $n, k \in \mathbf{R}$, show that $\dfrac{d^2y}{dx^2} + 2n\dfrac{dy}{dx} + (n^2 + k^2)y = 0$.

Differentiation of Natural Logarithmic Functions

Logarithms to the base e are called '**natural logarithms**'.
The notation $\ln x$ is used as an abbreviation of $\log_e x$.
The function $y = \ln x$ is the inverse function of $y = e^x$
(exponents and logs are inverse functions of each other).

Note: $\log_e x$ or $\ln x$ is defined only for $x > 0$.

Natural logarithms obey the same laws as logarithms to any other base.

Laws of Logs:	$\ln ab = \ln a + \ln b$	$\ln \frac{a}{b} = \ln a - \ln b$	$\ln a^n = n \ln a$

Using the laws of logs before differentiating can simplify the work.

The following is worth remembering when evaluating the derivatives of natural logarithmic functions:

$$\ln e^k = k, \qquad \text{for any } k \in \mathbf{R}.$$

For example,

$\ln 1 = \ln e^0 = 0,$ $\quad \ln e = \ln e^1 = 1,$ $\quad \ln e^2 = 2,$ $\quad \ln\sqrt{e} = \ln e^{1/2} = \frac{1}{2}.$

The rules for differentiating also apply to natural logarithmic functions.

Suppose u is a function of x.

If $\quad y = \ln u$

then $\quad \dfrac{dy}{dx} = \dfrac{1}{u} \cdot \dfrac{du}{dx}.$

Example

Find $\dfrac{dy}{dx}$ if **(i)** $y = \ln(x^2 + 1)$ **(ii)** $y = \ln(\sin x)$ **(iii)** $y = \ln\sqrt{x^2 - 3}$ **(iv)** $y = x \ln x$.

Solution:

(i) $y = \ln(x^2 + 1)$

$$\frac{dy}{dx} = \frac{1}{x^2+1} \cdot 2x$$

$$= \frac{2x}{x^2+1}$$

(ii) $y = \ln(\sin x)$

$$\frac{dy}{dx} = \frac{1}{\sin x} \cdot \cos x$$

$$= \frac{\cos x}{\sin x} = \cot x$$

(iii) $y = \ln\sqrt{x^2 - 3}$

$y = \ln(x^2 - 3)^{1/2} = \frac{1}{2} \ln(x^2 - 3)$

(using $\ln a^n = n \ln a$)

$$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{x^2-3} \cdot 2x$$

$$= \frac{x}{x^2-3}$$

(iv) $y = x \ln x$

$$\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \ln(x)(1)$$

(using the product rule)

$$= 1 + \ln x$$

Exercise 12.8

Find $\frac{dy}{dx}$ for each of the following:

1. $y = \ln 5x$
2. $y = \ln(2x + 3)$
3. $y = \ln(x^2 + 3$
4. $y = \ln(\cos x)$
5. $y = \ln\left(\frac{1}{x}\right)$
6. $y = \ln(e^x + 2$
7. $y = \ln(\sin 2x)$
8. $y = \ln(\tan 3x)$
9. $y = \ln(e^{2x})$
10. $y = x \ln x^2$
11. $y = x^3 \ln(x + 1)$
12. $y = x^2 \ln 4x$

Use the rules of logarithms, or otherwise, to find $\frac{dy}{dx}$ for each of the following:

13. $y = \ln\left(\frac{2x}{x+1}\right)$
14. $y = \ln(2x + 3)^2$
15. $y = \ln\left(\frac{1}{e^x}\right)$
16. $y = \ln\sqrt{1 + x^2}$
17. $y = \ln\sqrt{\sin x}$
18. $y = \ln\sqrt{\frac{x}{1+}}$
19. If $f(x) = \ln(e^x \cos x)$, show that $f'(x) = 1 - \tan x$.
20. If $y = \ln(\sec x + \tan x)$, show that $\frac{dy}{dx} = \sec x$.
21. If $f(x) = x^2 \ln x$, evaluate **(i)** $f'(e)$ **(ii)** $f'(1)$.
22. Given $f(x) = \ln\left(\frac{1 + \cos x}{1 - \cos x}\right)$, show that $f'(x) = -2 \operatorname{cosec} x$.
23. If $f(x) = \ln(\ln x)$, evaluate $f'(e)$.
24. If $f(x) = \ln\left(\frac{e^x}{1 + e^x}\right)$ evaluate $f'(0)$.
25. If $y = \frac{\ln x}{x}$, show that $\frac{dy}{dx} = \frac{1 - \ln x}{x^2}$.

    Evaluate $\frac{d^2y}{dx^2}$ at $x = e$.
26. Given $f(x) = e^x \ln x$, $x > 0$, evaluate $f''(1)$.
27. Given $y = \ln(t + 1)$ and $x = 1 + \ln t$, express $\frac{dy}{dx}$ in terms of t.
28. If $y = e^{t+1}$ and $x = e^t$, find the value of $\ln\left(\frac{dy}{dx}\right)$.
29. If $y = \ln t$ and $x + \frac{1}{2}\left(t + \frac{1}{t}\right)$, show that $\frac{dy}{dx} = \frac{2t}{t^2 - 1}$.
30. If $x = \ln\left(\frac{e^t}{1 + e^t}\right)$ and $y = \ln\left(\frac{1 + e^t}{e^t}\right)$, $t \in \mathbf{R}$, evaluate $\frac{dy}{dx}$.
31. Given $y = x \ln(x^2)$, show that $x\left(\frac{dy}{dx}\right) - 2x = y$.

32. Using $\ln \frac{a}{b} = \ln a - \ln b$, or otherwise, show that if $y = \ln\left(\frac{1+x}{1-x}\right)$,

(i) $(1-x^2)\frac{dy}{dx} = 2$ **(ii)** $\left(\frac{2x}{1-x^2}\right)\frac{dy}{dx} - \frac{d^2y}{dx^2} = 0.$

33. Factorise $a^x + a^{2x}$. If $y = \ln(1+e^x)$, show that $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = \frac{dy}{dx}$.

34. If $y = \ln e^{-x}\sqrt{\frac{1+2x}{1-2x}}$, show that $\frac{dy}{dx} = \frac{1+4x^2}{1-4x^2}$.

Find the value of $\frac{dy}{dx}$ at $x = -1$.

(Hint: $\ln \frac{ab}{c} = \ln a + \ln b - \ln c$)

Logarithmic Differentiation

Functions of the form 2^x, x^x or $3^{\sin x}$ are differentiated using '**logarithmic differentiation**'.

The method involves three steps:

1. Take natural logs of both sides and use the fact that $\ln a^x = x \ln a$.
2. Differentiate both sides with respect to x, using implicit differentiation.
3. Multiply both sides by y to get $\frac{dy}{dx}$ on its own.

Example

Differentiate **(i)** 2^x **(ii)** x^x with respect to x.

Solution:

(i) Let

$$y = 2^x$$
$$\ln y = \ln 2^x$$
$$\ln y = x \ln 2$$
$$\frac{1}{y}\frac{dy}{dx} = \ln 2$$
$$\frac{dy}{dx} = y \ln 2$$
$$= 2^x \ln 2$$

(ii) Let

$$y = x^x$$
$$\ln y = \ln x^x$$
$$\ln y = x \ln x$$
$$\frac{1}{y}\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \ln x(1)$$

(using the product rule)

$$\frac{1}{y}\frac{dy}{dx} = 1 + \ln x$$
$$\frac{dy}{dx} = y(1 + \ln x)$$
$$= x^x(1 + \ln x)$$

Exercise 12.9

Use logarithmic differentiation to find the derivative of each of the following:

1. 3^x
2. 5^x
3. 3^{2x}
4. 4^{3x+1}
5. $2^{\sin x}$
6. $2^{\ln x}$
7. $(\sin x)^x$
8. $2^x x^2$
9. If $f(x) = x4^x$, evaluate $f'(1)$.
10. If $y = a^x$, $a > 0$, $a \in \mathbf{R}$, show that $\dfrac{dy}{dx} = a^x \ln a$.
11. If $x^y = e^x$, show that $\dfrac{dy}{dx} = \dfrac{x-y}{x \ln x}$ or $\dfrac{\ln x - 1}{(\ln x)^2}$.