Equation of L [slope $= -\frac{1}{2}$, point $= (-3, -2)$]:	Equation of M [slope $= -3$, point $= (-\frac{3}{2}, -\frac{3}{2})$]:
$(y - y_1) = m(x - x_1)$	$(y - y_1) = m(x - x_1)$
$(y + 2) = -\frac{1}{2}(x + 3)$	$(y + \frac{3}{2}) = -3(x + \frac{3}{2})$
$2y + 4 = -x - 3$	$y + \frac{3}{2} = -3x - \frac{9}{2}$
L: $x + 2y + 7 = 0$	M: $3x + y + 6 = 0$

Solving the simultaneous equations L and M gives the centre of the circle $c(-1, -3)$.

Radius:

The radius is $|ac|$ or $|bc|$.

$a(-3, -2)$, $c(-1, -3)$

$|ac| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(-1 + 3)^2 + (-3 + 2)^2} = \sqrt{(2)^2 + (-1)^2} = \sqrt{4 + 1} = \sqrt{5}$

Thus, the centre of the circle is $(-1, -3)$ and the radius is $\sqrt{5}$.

Equation of the circle:
$$(x - h)^2 + (y - k)^2 = r^2$$
$$(x + 1)^2 + (y + 3)^2 = (\sqrt{5})^2$$
$$(x + 1)^2 + (y + 3)^2 = 5$$
or
$$x^2 + y^2 + 2x + 6y + 5 = 0$$

Note: After finding the equation of the line L, we could have let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ and used an algebraic approach to find the values of g, f and c.

Exercise 1.8

1. Find the equation of the circle that contains the points (3, 6) and (5, 4) and whose centre lies on the line $x + y - 5 = 0$.

2. Find the equation of the circle that contains the points (2, −6) and (4, −2) and whose centre lies on the line $2x + y + 4 = 0$.

3. The circle $x^2 + y^2 + 2gx + 2fy + c = 0$ passes through the points (4, 3) and (6, −3).
The line $3x - y - 7 = 0$ passes through the centre of the circle.
Find the real numbers g, f and c.

4. Find the equation of the circle which passes through the points (−1, 6) and (−3, 0) and where the line $2x + y + 6 = 0$ is a tangent at the point (−3, 0).

5. Find the equation of the circle which passes through the points (−1, 3) and (3, 5) and where the line $3x - y + 6 = 0$ is a tangent at the point (−1, 3).

6. The circle $x^2 + y^2 + 2gx + 2fy + c = 0$ passes through the points (4, 1) and (6, −5).
The line $2x - y - 17 = 0$ is a tangent to the circle at (6, −5).
Find the real numbers g, f and c.

Given the length of the radius

In some questions we are given the radius. When this happens we let $\sqrt{g^2+f^2-c}$ be equal to the given radius. Then we square both sides. We then have to use the other information in the question to form two other equations in g, f and c, and substitute these into the first equation to get a quadratic equation in one variable. In general, we end up with two circles that satisfy the given conditions.

Example

A circle of radius length $\sqrt{20}$ contains the point (–1, 3). Its centre lies on the line $x+y=0$. Find the equations of the two circles that satisfy these conditions.

Solution:

Let the circle be $x^2+y^2+2gx+2fy+c=0$.

Given: Radius $=\sqrt{20}$

$\therefore \quad \sqrt{g^2+f^2-c}=\sqrt{20}$

$g^2+f^2-c=20$ ①

Contains the point (–1, 3)

$\therefore (-1)^2+(3)^2+2g(-1)+2f(3)+c=0$

$2g-6f-c=10$ ②

The centre $(-g, -f)$ is on the line $x+y=0$

$\therefore \quad -g-f=0$

$g+f=0$ ③

We now have to solve between the simultaneous equations ①, ② and ③.

$g+f=0$ ③

$f=-g$

Put $f=-g$ into ② and express c in terms of g only.

$2g-6f-c=10$ ②

$2g-6(-g)-c=10 \quad (f=-g)$

$2g+6g-c=10$

$8g-c=10$

$c=8g-10$

We now have f and c in terms of g and we put these into ①:

$g^2+f^2-c=20$ ①

$g^2+(-g)^2-(8g-10)=20$ (put in $f=-g$ and $c=8g-10$)

$g^2+g^2-8g+10=20$

$2g^2-8g-10=0$

$g^2-4g-5=0$

$(g+1)(g-5)=0$

$g=-1$ or $g=5$

Case 1

$g=-1$

$f=-g=-(-1)=1$

$c=8g-10=8(-1)-10=-8-10=-18$

$x^2+y^2+2(-1)x+2(1)y-18=0$

$x^2+y^2-2x+2y-18=0$

Case 2

$g=5$

$f=-g=-5$

$c=8g-10=8(5)-10=40-10=30$

$x^2+y^2+2(5)x+2(-5)y+30=0$

$x^2+y^2+10x-10y+30=0$

These are the equations of the two circles that satisfy the given conditions.

Exercise 1.9

1. A circle of radius length 2 contains the point $(1, -1)$. Its centre lies on the line $x = 1$. Find the equations of the two circles that satisfy these conditions.

2. A circle of radius length $\sqrt{20}$ contains the point $(0, 2)$. Its centre lies on the line $x + y = 0$. Find the equations of the two circles that satisfy these conditions.

3. A circle of radius length $\sqrt{10}$ contains the point $(-5, 0)$. Its centre lies on the line $x + 2y = 0$. Find the equations of the two circles that satisfy these conditions.

4. A line L: $2x - 3y = 0$ is a tangent to a circle C at the point $(0, 0)$. If the radius of C is $\sqrt{13}$, find two possible equations for C.

5. Two circles intersect at the points $p(1, 3)$ and $q(3, -1)$. The line M joining the centres of the circles is the perpendicular bisector of $[pq]$.
   (i) Find the coordinates of r, the midpoint of $[pq]$ and the equation of the line M.
   (ii) If the distance from the centre of each circle to r is $\sqrt{20}$, find the radius length of each circle.
   (iii) Find the equation of each circle.

6. A circle of radius length $\sqrt{10}$ contains the points $(1, 2)$ and $(-1, 4)$. Find the equations of the two circles that satisfy these conditions.

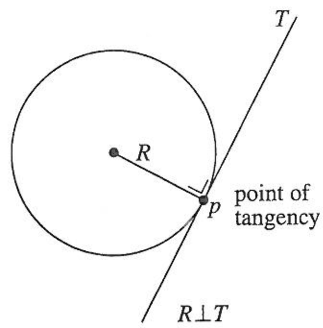
Equation of a Tangent I

Equation of a tangent to a circle at a given point

A tangent is perpendicular to the radius that joins the centre of a circle to the point of tangency.

This fact is used to find the slope of the tangent.

In the diagram on the right, the radius, R, is perpendicular to the tangent, T, at the point of tangency, p.



The equation of a tangent to a circle at a given point is found with the following steps:

Step 1: Find the slope of the radius to the point of tangency.

Step 2: Turn this slope upside down and change its sign. This gives the slope of the tangent.

Step 3: Use the coordinates of the point of contact and the slope of the tangent at this point in the formula:

$$(y - y_1) = m(x - x_1).$$

This gives the equation of the tangent.

A diagram is often very useful.

Example

Find the equation of the tangent to the circle $x^2+y^2-4x+6y-12=0$ at the point $(5,-7)$ on the circle.

Solution:

$x^2+y^2-4x+6y-12=0$

The centre of the circle is $(2,-3)$.

Slope of $R=\dfrac{-7+3}{5-2}=-\dfrac{4}{3}$

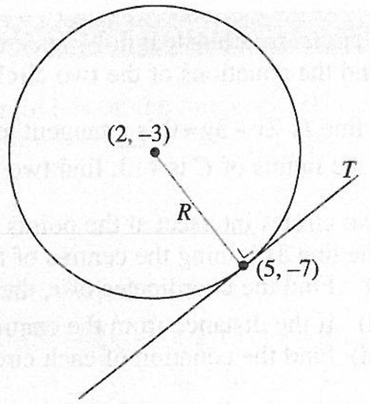
$\therefore$ Slope of $T=\dfrac{3}{4}$

Equation of T: $(y-y_1)=m(x-x_1)$

$(y+7)=\frac{3}{4}(x-5)$

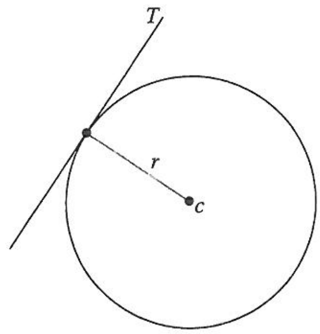
$4y+28=3x-15$

$3x-4y-43=0$



Proving that a line is a tangent to a circle

A line is a tangent to a circle if the perpendicular distance from the centre of the circle to the line is equal to the radius.



Example

Prove that the line $2x-3y-27=0$ is a tangent to the circle $x^2+y^2-8x+4y+7=0$.

Solution:

(As we do not need the point of contact, we use the perpendicular distance method.)

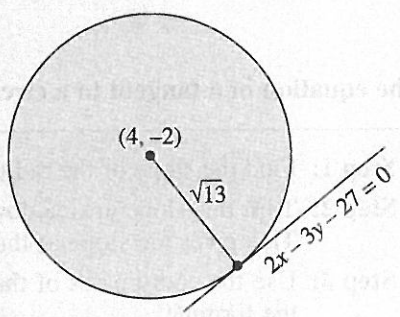
$x^2+y^2-8x+4y+7=0$

Centre $=(4,-2)$

Radius $=\sqrt{(4)^2+(-2)^2-7}=\sqrt{16+4-7}=\sqrt{13}$

Perpendicular distance from the centre $(4,-2)$ to the line $2x-3y-27=0$ is given by:

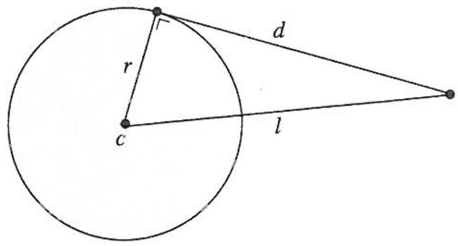
$$d=\frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}=\frac{|2(4)-3(-2)-27|}{\sqrt{(2)^2+(-3)^2}}=\frac{|8+6-27|}{\sqrt{4+9}}=\frac{|-13|}{\sqrt{13}}=\frac{13}{\sqrt{13}}=\sqrt{13}$$



As the perpendicular distance from the centre of the circle to the line is equal to the radius, the line is a tangent to the circle.

Length of a tangent to a circle from a point outside the circle

The **length of a tangent** from a point outside a circle is the distance, d, from the point outside the circle to the point of tangency.



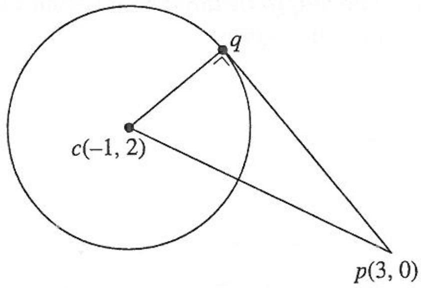
Method:

1. Find the centre, c, and radius length, r, of the circle.
2. Find the distance, l, between the centre and the point outside the circle.
3. Use Pythagoras's theorem to find d, i.e. $l^2 = r^2 + d^2$.

Example

A tangent from the point $p(3, 0)$ touches the circle $x^2 + y^2 + 2x - 4y + 1 = 0$ at q.
Find $|pq|$.

(rough diagram)



Solution:

$x^2 + y^2 + 2x - 4y + 1 = 0$

centre $= c(-1, 2)$

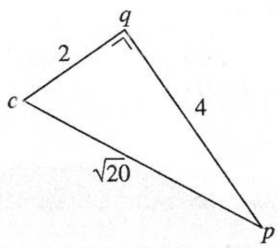
radius $= \sqrt{(-1)^2 + (2)^2 - 1} = \sqrt{1 + 4 - 1} = \sqrt{4} = 2$

Distance from the centre $c(-1, 2)$ to the point $p(3, 0)$

$= |pc| = \sqrt{(3+1)^2 + (0-2)^2} = \sqrt{4^2 + (-2)^2} = \sqrt{16 + 4} = \sqrt{20}$

Using Pythagoras's theorem:

$$|pq|^2 + |qc|^2 = |pc|^2$$
$$|pq|^2 + 2^2 = (\sqrt{20})^2$$
$$|pq|^2 + 4 = 20$$
$$|pq|^2 = 16$$
$$|pq| = 4$$



Exercise 1.10

Find the equation of the tangent to the given circle at the given point:

1. $x^2 + y^2 = 10$; $(3, 1)$
2. $x^2 + y^2 = 29$; $(-2, 5)$
3. $x^2 + y^2 = 50$; $(-7, -1)$
4. $4x^2 + 4y^2 = 25$; $\left(2, -\frac{3}{2}\right)$
5. $(x-4)^2 + (y+3)^2 = 10$; $(7, -4)$
6. $x^2 + (y-5)^2 = 29$; $(5, 3)$
7. $x^2 + y^2 + 6x + 2y - 3 = 0$; $(-5, -4)$
8. $x^2 + y^2 - 2x + 4y - 15 = 0$; $(3, 2)$
9. $x^2 + y^2 - 4x - 6y - 12 = 0$; $(5, 7)$
10. $x^2 + y^2 - 8x + 14 = 0$; $(3, -1)$

11. Show that the tangents to the circle $x^2 + y^2 - 2x - 2y - 3 = 0$ at the points $(3, 2)$ and $(2, -1)$ are perpendicular to each other.

Verify in each case that the line L is a tangent to the circle C:

12. L: $3x + 4y - 25 = 0$; C: $x^2 + y^2 = 25$

13. L: $4x + 3y - 14 = 0$; C: $x^2 + y^2 - 4x + 6y + 4 = 0$

14. L: $x - 4y + 31 = 0$; C: $x^2 + y^2 + 4x - 6y - 4 = 0$

15. L: $x - 6y - 9 = 0$; C: $x^2 + y^2 - 4x - 10y - 8 = 0$

16. L: $x - 2y + 10 = 0$; C: $x^2 + y^2 + 8x - 16y + 60 = 0$

In each of the following, find the distance from the given point outside the circle to the point of tangency:

17. $(11, -2)$; $x^2 + y^2 = 25$

18. $(0, -4)$; $x^2 + y^2 - 6x - 8y + 16 = 0$

19. $(3, 1)$; $x^2 + y^2 + 4x - 2y - 4 = 0$

20. $(2, 5)$; $x^2 + y^2 - 2x + 4y - 20 = 0$

21. $(0, 0)$; $x^2 + y^2 - 8x - 6y + 20 = 0$

22. $(5, 0)$; $x^2 + y^2 + 6x - 5 = 0$

23. The length of the tangent from the point $(3, 2)$ to the circle $x^2 + y^2 - 8x - 8y + k = 0$ is 2. Find the value of k.

Equation of a Tangent 2

Tangents parallel, or perpendicular, to a given line

We make use of the fact that the perpendicular distance from the centre of a circle to the tangent is equal to the radius.

Note: $ax + by + k = 0$ is a line parallel to the line $ax + by + c = 0$.
$bx - ay + k = 0$ or $-bx + ay + k = 0$ is a line perpendicular to the line $ax + by + c = 0$.

Example

Find the equations of the tangents to the circle $x^2 + y^2 - 6x - 2y - 15 = 0$ which are parallel to the line $3x + 4y + 20 = 0$.

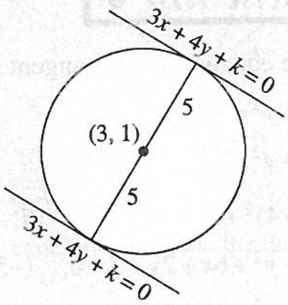
Solution:

$x^2 + y^2 - 6x - 2y - 15 = 0$

Centre $= (3, 1)$

Radius $= \sqrt{(3)^2 + (1)^2 + 15} = \sqrt{9 + 1 + 15} = \sqrt{25} = 5$

Let the equation of the tangent parallel to $3x + 4y + 20 = 0$ be $3x + 4y + k = 0$.



As $3x+4y+k=0$ is a tangent to the circle, the perpendicular distance from the centre of the circle, (3, 1), to this line is equal to the radius, 5.

$$\therefore \quad \frac{|3(3)+4(1)+k|}{\sqrt{(3)^2+(4)^2}}=5$$

$$\frac{|9+4+k|}{\sqrt{9+16}}=5$$

$$\frac{|13+k|}{5}=5$$

$$|13+k|=25$$

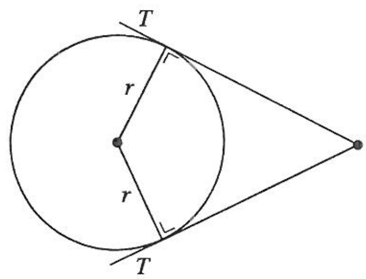
$$\therefore \quad 13+k=25 \quad \text{or} \quad 13+k=-25$$

$$k=12 \quad \text{or} \quad k=-38$$

Thus, the tangents are $3x+4y+12=0$ and $3x+4y-38=0$.

Equations of tangents from a point outside a circle

From a point outside a circle, two tangents can be drawn to touch the circle.



Method for finding the two equations of tangents from a point (x_1, y_1) outside a circle:

1. Find the centre and radius length of the circle (a rough diagram can help).
2. Let the equation be $(y-y_1)=m(x-x_1)$ and write the equation in the form $ax+by+c=0$.
3. Let the perpendicular distance from the centre of the circle to the tangent equal the radius.
4. Solve this equation to find two values of m.
5. Using these two values of m and the point (x_1, y_1), write down the equations of the two tangents.

Example

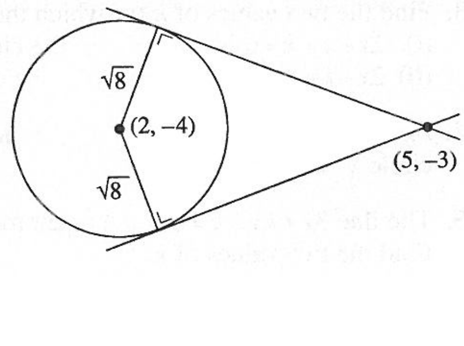
Find the equations of the two tangents from the point (5, –3) to the circle $x^2+y^2-4x+8y+12=0$.

Solution:

$x^2+y^2-4x+8y+120$

Centre $=(2, -4)$

Radius $=\sqrt{(2)^2+(-4)^2-12}=\sqrt{4+16-12}=\sqrt{8}$



We have a point $(5, -3)$ and we need to find the slopes of the two tangents.

Equation: $(y - y_1) = m(x - x_1)$

$$(y + 3) = m(x - 5)$$
$$y + 3 = mx - 5m$$
$$mx - y + (-5m - 3) = 0 \qquad \text{[in the form } ax + by + c = 0\text{]}$$

The perpendicular distance from the centre of the circle $(2, -4)$ to each tangent is equal to the radius, $\sqrt{8}$.

Thus,
$$\frac{|m(2) - 1(-4) + (-5m - 3)|}{\sqrt{m^2 + (-1)^2}} = \sqrt{8}$$
$$\frac{|2m + 4 - 5m - 3|}{\sqrt{m^2 + 1}} = \sqrt{8}$$
$$\frac{|-3m + 1|}{\sqrt{m^2 + 1}} = \sqrt{8}$$
$$\frac{9m^2 - 6m + 1}{m^2 + 1} = 8 \qquad \text{[square both sides]}$$
$$9m^2 - 6m + 1 = 8m^2 + 8 \qquad \text{[multiply both sides by } (m^2 + 1)\text{]}$$
$$m^2 - 6m - 7 = 0$$
$$(m + 1)(m - 7) = 0$$
$$m = -1 \quad \text{or} \quad m = 7$$

Equations of the two tangents:

Slope $= -1$; point $= (5, -3)$
$$(y + 3) = -1(x - 5)$$
$$y + 3 = -x + 5$$
$$x + y - 2 = 0$$

Slope $= 7$; point $= (5, -3)$
$$(y + 3) = 7(x + 5)$$
$$y + 3 = 7x + 35$$
$$7x - y + 32 = 0$$

Thus, the equations of the two tangents are $x + y - 2 = 0$ and $7x - y + 32 = 0$.

Exercise 1.11

1. The line $2x + y + k = 0$ is a tangent to the circle $x^2 + y^2 = 5$. Find the two values of k.

2. The line $3x + y + k = 0$ is a tangent to the circle $x^2 + y^2 - 4x + 8y + 10 = 0$. Find the two values of k.

3. Find the two values of k for which the line:
   (i) $2x + y + k = 0$ is a tangent to the circle $x^2 + y^2 - 4x - 6y + 8 = 0$
   (ii) $2x + ky + 3 = 0$ is a tangent to the circle $x^2 + y^2 - 4x - 4y - 5 = 0$.

4. Find the two values of k for which the line $x + ky - 6 = 0$ is a tangent to the circle $x^2 + y^2 - 6x - 2y + 5 = 0$.

5. The line $3x + ky - k = 0$ is a tangent to the circle $x^2 + y^2 - 10x - 2y + 17 = 0$. Find the two values of k.

6. Find the equations of the tangents to the circle $x^2 + y^2 = 25$ which are parallel to the line $4x - 3y + 10 = 0$.

7. Find the equations of the tangents to the circle $x^2 + y^2 = 10$ which are parallel to the line $3x - y = 0$.

8. Find the equations of the tangents to the circle $x^2 + y^2 + 6x + 10y + 29 = 0$ which are parallel to the line $2x - y - 8 = 0$.

9. Find the equations of the tangents to the circle $x^2 + y^2 - 6x + 4y - 4 = 0$ which are parallel to the line $x + 4y - 3 = 0$.

10. Find the equations of the tangents to the circle $x^2 + y^2 + 6x - 2y - 15 = 0$ which are perpendicular to the line $4x + 3y + 5 = 0$.

11. The line $mx - y = 0$ is a tangent to the circle $x^2 + y^2 - 6x + 2y + 2 = 0$.
Find the two values of m.

Find the equations of the tangent from the given point to the given circle:

12. $(5, 0)$; $x^2 + y^2 = 5$

13. $(-10, -10)$; $x^2 + y^2 = 20$

14. $(0, 0)$; $x^2 + y^2 + 4x + 2y + 4 = 0$

15. $(3, -2)$; $x^2 + y^2 + 4x - 6y + 8 = 0$

16. $(0, 1)$; $x^2 + y^2 - 8x - 2y + 9 = 0$

17. $(1, 2)$; $x^2 + y^2 + 8x + 6y + 15 = 0$

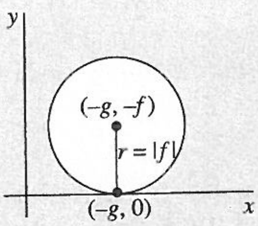
18. The line $ax + by = 0$ is a tangent to the circle $x^2 + y^2 - 4x - 2y + 4 = 0$, where $a, b \in \mathbf{R}$ and $b \neq 0$.
   (i) Show that $\frac{a}{b} = -\frac{4}{3}$.
   (ii) Hence, or otherwise, find the coordinates of the point of contact.

Circles with the Axes as Tangents

If a circle touches an axis (the x- or y-axis is a tangent to the circle), then one of the coordinates of the centre of the circle is equal to the radius.

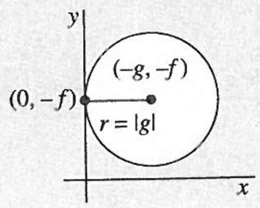
1. Circle touching the x-axis

$$\text{radius} = |-f|$$
$$\sqrt{g^2 + f^2 - c} = |-f|$$
$$g^2 + f^2 - c = f^2$$
$$g^2 - c = 0$$
$$g^2 = c \leftarrow \text{constant}$$



2. Circle touching the y-axis

$$\text{radius} = |-g|$$
$$\sqrt{g^2 + f^2 - c} = |-g|$$
$$g^2 + f^2 - c = g^2$$
$$f^2 - c = 0$$
$$f^2 = c \leftarrow \text{constant}$$



Example

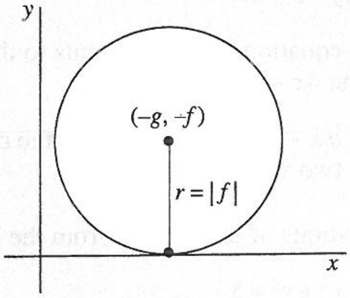
The x-axis is a tangent to the circle C: $x^2+y^2+2gx+2fy+c=0$.
Show that $g^2=c$.
The points (2, 1) and (3, 2) are also on the circle C.
Find two possible equations for C.

Solution:

$x^2+y^2+2gx+2fy+c=0$

centre $=(-g,-f)$ radius $=\sqrt{g^2+f^2-c}$

From the diagram,



$$\text{radius}=|-f|$$
$$\sqrt{g^2+f^2-c}=|-f|$$
$$g^2+f^2-c=f^2$$
$$g^2-c=0$$
$$g^2=c$$

(2, 1) on the circle: $(2)^2+(1)^2+2g(2)+2f(1)+c=0 \Rightarrow 4g+2f+c+5=0$

(3, 2) on the circle: $(3)^2+(2)^2+2g(3)+2f(2)+c=0 \Rightarrow 6g+4f+c+13=0$

We now have 3 equations:

$g^2=c$ ① $\quad 4g+2f+c+5=0$ ② $\quad 6g+4f+c+13=0$ ③

Replace c with g^2 in ② and ③:

$4g+2f+c+5=0$ ② $\quad 6g+4f+c+13=0$ ③

$4g+2f+g^2+5=0$ ④ $\quad 6g+4f+g^2+13=0$ ⑤

From ④, express f in terms of g and put this into ⑤:

$$4g+2f+g^2+5=0 \quad ④$$
$$2f=-g^2-4g-5$$
$$f=\left(\frac{-g^2-4g-5}{2}\right)$$

put this into ⑤.

$$6g+4f+g^2+13=0 \quad ⑤$$
$$6g+4\left(\frac{-g^2-4g-5}{2}\right)+g^2+13=0$$
$$6g-2g^2-8g-10+g^2+13=0$$
$$-g^2-2g+3=0$$
$$g^2+2g-3=0$$
$$(g+3)(g-1)=0$$
$$g=-3 \quad \text{or} \quad g=1$$

Case 1

$g=-3$

$c=g^2=(-3)^2=9$

$$f=\frac{-g^2-4g-5}{2}=\frac{-9+12-5}{2}=\frac{-2}{2}=-1$$
$$x^2+y^2+2(-3)x+2(-1)y+9=0$$
$$x^2+y^2-6x-2y+9=0$$

Case 2

$g=1$

$c=g^2=(1)^2=1$

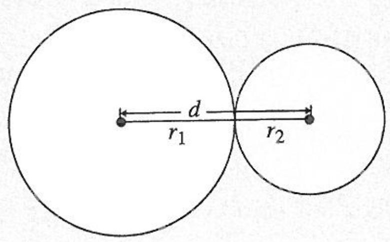
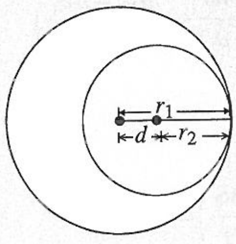
$$f=\frac{-g^2-4g-5}{2}=\frac{-1-4-5}{2}=\frac{-10}{2}=-5$$
$$x^2+y^2+2(1)x+2(-5)y+1=0$$
$$x^2+y^2+2x-10y+1=0$$

Exercise 1.12

1. Find the equation of the circle with centre (2, 5) and which has the x-axis as a tangent.
2. Find the equation of the circle with centre (–3, 4) and which has the y-axis as a tangent.
3. Show that the circle $x^2+y^2-4x-4y+4=0$ touches the x- and y-axes.
4. The x-axis is a tangent to the circle $x^2+y^2+2gx+2fy+c=0$.
   Show that $g^2=c$.
   The x-axis is a tangent to a circle C at the point (5, 0).
   The point (1, 4) is on C. Find the equation of C.
5. The y-axis is a tangent to the circle $x^2+y^2+2gx+2fy+c=0$.
   Show that $f^2=c$.
   The y-axis is a tangent to a circle K at the point (0, 2).
   The point (2, –2) is on K. Find the equation of K.
6. A circle has its centre in the first quadrant and touches the x- and y-axes.
   If the distance from the centre of the circle to the origin is $2\sqrt{2}$, find the equation of the circle.
7. The x-axis and the y-axis are tangents to the circle $x^2+y^2+2gx+2fy+c=0$.
   Show that: **(i)** $g^2=c$ **(ii)** $g^2=f^2$.
   The circle C: $x^2+y^2+2gx+2fy+c=0$ has its centre in the first quadrant.
   The x- and y-axes are tangents to C. The point (3, 6) is on C.
   Find two equations for C.
8. Show that the circle $x^2+y^2-2rx-2ry+r^2=0$ has radius r and has the x- and y-axes as tangents.
   Find the equations of the two circles which contain the point (1, 2) and have the x- and y-axes as tangents.

Touching Circles

Two circles are said to be **touching** if they have only one point of intersection. To investigate whether two circles touch, we compare the distance between their centres with the sum or difference of their radii.
Consider two circles of radius r_1 and r_2 (where $r_1 > r_2$) and let d be the distance between their centres.

1. Circles touch externally	**2. Circles touch internally**
	
$d=r_1+r_2$ Distance between their centres = sum of their radii	$d=r_1-r_2$ Distance between their centres = difference of their radii

Example

S: $x^2+y^2-16y+32=0$ and K: $x^2+y^2-18x+2y+32=0$ are two circles.
Show that the circles touch externally and find their point of contact.

Solution:

S: $x^2+y^2+0x-16y+32=0$

centre $=(0, 8)=c_1$

radius $=\sqrt{0^2+8^2-32}$

$=\sqrt{64-32}=\sqrt{32}=4\sqrt{2}$

K: $x^2+y^2-18x+2y+32=0$

centre $=(9, -1)=c_2$

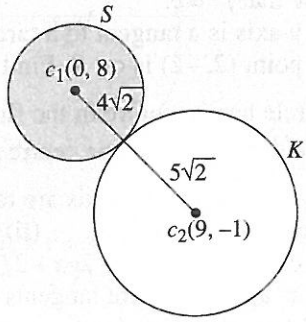
radius $=\sqrt{9^2+(-1^2)-32}$

$=\sqrt{81+1-32}=\sqrt{50}=5\sqrt{2}$

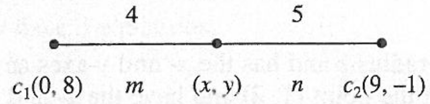
$r_1+r_2=4\sqrt{2}+5\sqrt{2}=9\sqrt{2}$

distance between centres

$=\sqrt{(9-0)^2+(-1-8)^2}=\sqrt{81+81}=\sqrt{162}=9\sqrt{2}$



Thus, the circles touch externally, as $r_1+r_2=|c_1c_2|$.
To determine the point of contact, divide the line segment joining the centres in the ratio 4:5.
Let the point of contact be (x, y).



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Method 1:

$$\frac{x-0}{9-x}=\frac{4}{5}$$

$$5x=36-4x$$

$$9x=36$$

$$x=4$$

$$\frac{y-8}{-1-y}=\frac{4}{5}$$

$$5y-40=-4-4x$$

$$9y=36$$

$$y=4$$

Thus, the point of contact is (4, 4).

Method 2: Using the formula,

$$(x, y)=\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$$

$$=\left(\frac{4(9)+5(0)}{4+5}, \frac{4(-1)+5(8)}{4+5}\right)$$

$$=\left(\frac{36}{9}, \frac{36}{9}\right)=(4, 4)$$

Thus, the point of contact is (4, 4).

Exercise 1.13

1. Prove that the circles $x^2 + y^2 + 2x + 2y - 7 = 0$ and $x^2 + y^2 - 6x - 4y + 9 = 0$ touch externally.

2. Prove that the circles $x^2 + y^2 + 6x + 16y + 9 = 0$ and $x^2 + y^2 - 4x - 8y - 5 = 0$ touch externally.

3. Prove that the circles $x^2 + y^2 + 12x - 6y - 76 = 0$ and $x^2 + y^2 - 4x + 6y + 12 = 0$ touch internally.

4. Prove that the circles $x^2 + y^2 = 80$ and $x^2 + y^2 - 12x - 6y + 40 = 0$ touch internally.

5. Prove that the circles $x^2 + y^2 - 2x - 4y - 20 = 0$ and $x^2 + y^2 - 18x - 16y + 120 = 0$ touch externally and find their point of contact.

6. Prove that the circles $x^2 + y^2 + 14x - 10y - 26 = 0$ and $x^2 + y^2 - 4x + 14y + 28 = 0$ touch externally and find their point of contact.

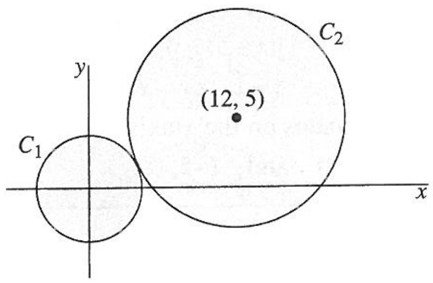
7. Prove that the circles $x^2 + y^2 - 6x + 4y + 11 = 0$ and $x^2 + y^2 + 4x - 6y - 19 = 0$ touch externally and find their point of contact.

8. Prove that the circles $x^2 + y^2 + 4x + 6y - 19 = 0$ and $x^2 + y^2 - 2x - 1 = 0$ touch internally and find their point of contact.

9. Prove that the circles $x^2 + y^2 + 8x - 8y + 24 = 0$ and $x^2 + y^2 + 2x - 2y = 0$ touch externally and find their point of contact.

10. The diagram shows the circle C_1: $x + y^2 = 16$ and the circle C_2 with centre (12, 5).

    If C_1 and C_2 touch externally, find the equation of C_2.

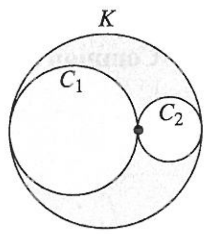


11. If the circles $x^2 + y^2 + 4x + 6y + k = 0$ and $x^2 + y^2 - 6x - 4y + 11 = 0$ touch externally, find the value of k.

12. C_1: $x^2 + y^2 - 6x - 4y - 3 = 0$ and

    C_2: $x^2 + y^2 - 18x - 4y + 81 = 0$ are two circles.

    (i) Prove that C_1 and C_2 touch externally.
    (ii) Find the point of contact of C_1 and C_2.
    (iii) K is a third circle.
    Both C_1 and C_2 touch K internally.
    Find the equation of K.

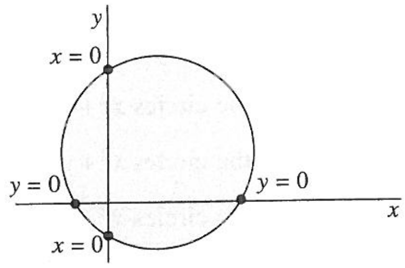


Note: All three centres lie on a straight line.

Chords of a Circle

Circles Intersecting the Axes

To find where a circle intersects the axes, we use the following:



The circle intersects the x-axis at $y = 0$.
The circle intersects the y-axis at $x = 0$.

Example

Find the coordinates of the points where the circle $x^2 + y^2 + 4x - 4y - 5 = 0$ intersects:
(i) the x-axis **(ii)** the y-axis.

Solution:

$$x^2 + y^2 + 4x - 4y - 5 = 0$$

(i) On the x-axis $y = 0$

(put in $y = 0$)

$$x^2 + 4x - 5 = 0$$
$$(x - 1)(x + 5) = 0$$
$$x = 1 \quad \text{or} \quad x = -5$$

Coordinates on the x-axis:

$(1, 0)$ and $(-5, 0)$

(ii) On the y-axis $x = 0$

(put in $x = 0$)

$$y^2 - 4y - 5 = 0$$
$$(y + 1)(y - 5) = 0$$
$$y = -1 \quad \text{or} \quad y = 5$$

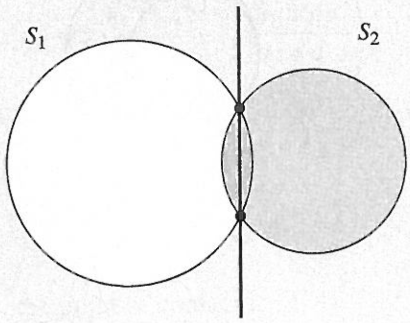
Coordinates on the y-axis:

$(0, -1)$ and $(0, 5)$

Common Chord or Common Tangent

If $S_1 = 0$ and $S_2 = 0$ are the equations of two circles in standard form, then $S_1 - S_2 = 0$ is the equation of the common chord or common tangent of the two circles.

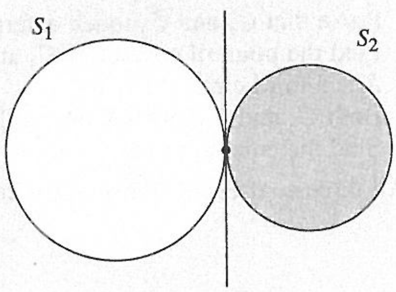
Common Chord



$S_1 - S_2 = 0$

Two points of intersection

Common Tangent



$S_1 - S_2 = 0$

One point of intersection

Note: To find the equation of the common chord, or common tangent, of two circles, $S_1=0$ and $S_2=0$, the coefficients of x^2 and y^2 must be the same for both circles. NB

To find the coordinates of the points of intersection of two circles, do the following:

1. Find the equation of the common chord ($S_1 - S_2 = 0$).
2. Solve between the equation of the common chord and the equation of one of the circles.

Example

The circles $x^2+y^2-6x+4y-7=0$ and $x^2+y^2-16x-6y+63=0$ intersect at the points p and q. Find the equation of the line pq.

Solution:

The equation of the line pq is given by: $S_1 - S_2 = 0$

$$(x^2+y^2-6x+4y-7)-(x^2+y^2-16x-6y+63)=0$$
$$x^2+y^2-6x+4y-7-x^2-y^2+16x+6y-63=0$$
$$10x+10y-70=0$$
$$x+y-7=0$$

Thus, the equation of the line pq (common chord) is $x+y-7=0$.

Note: To find the coordinates of p and q, solve between the equations point of intersection

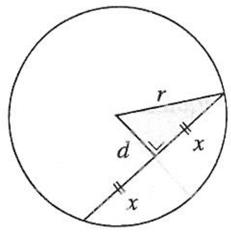
$x+y-7=0$ and $x^2+y^2-6x+4y-7=0$ or $x^2+y^2-16x-6y+63=0$.

Radius Perpendicular to a Chord

A radius (or part of a radius) that is perpendicular to a chord bisects that chord. This also enables us to use Pythagoras's Theorem:

$$d^2+x^2=r^2$$

Thus, knowing two of d, x and r, we can find the third.



Example

A circle K has centre (2, 3) and makes a chord of $8\sqrt{2}$ units on the y-axis. Find the equation of K.

Solution:

A rough diagram is very useful.

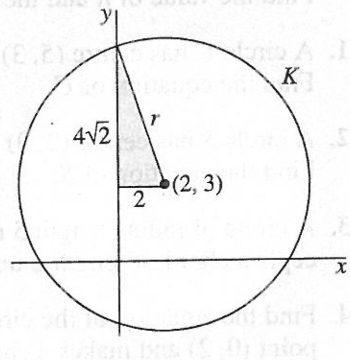
We have the centre and require the radius.

The length of the perpendicular from the centre (2, 3) to the y-axis is 2.

The length of the chord on the y-axis is $8\sqrt{2}$.

The perpendicular from the centre bisects the chord.

Thus, this length is $4\sqrt{2}$.



Using Pythagoras's theorem:

$$r^2 = (4\sqrt{2})^2 + (2)^2 = 32 + 4 = 36$$

$\therefore \quad r = 6$

Equation: $(x - h)^2 + (y - k)^2 = r^2$

$$(x - 2)^2 + (y - 3)^2 = 6^2$$

$$(x - 2)^2 + (y - 3)^2 = 36$$

or

$$x^2 + y^2 - 4x - 6y - 23 = 0$$

Exercise 1.14

1. Find the coordinates of the points where the circle $x^2 + y^2 - 6x + 6y - 16 = 0$ intersects:
(i) the x-axis **(ii)** the y-axis.

2. Find the length of the chord the y-axis makes with the circle $x^2 + y^2 - 8x - 10y + 16 = 0$.

3. Find the length of the chord the x-axis makes with the circle $x^2 + y^2 - 6x + 4y - 7 = 0$.

4. Find the equations of the tangents to the circle $x^2 + y^2 + 2x + 4y - 3 = 0$ at the points where the circle cuts the x-axis, and show that these tangents are perpendicular to each other.

5. The circles $x^2 + y^2 + 4x - 10y + 20 = 0$ and $x^2 + y^2 - 4x - 2y + 12 = 0$ intersect at the points a and b. Find the equation of the line ab.

6. The circles $x^2 + y^2 - 6x + 4 = 0$ and $x^2 + y^2 - 2x + 12y + 12 = 0$ intersect at the points p and q. Find the equation of the line pq.

7. The circles $x^2 + y^2 - 10x - 10y + 40 = 0$ and $x^2 + y^2 - 16x + 2y + 40 = 0$ intersect at the points a and b. Find the equation of the line ab, the coordinates of the point a and the coordinates of the point b.

8. The circles $x^2 + y^2 + 14x - 12y + 65 = 0$ and $x^2 + y^2 + 4x - 2y - 5 = 0$ intersect at the points a and b. Find the coordinates of the point a and the coordinates of the point b.

9. The circles $x^2 + y^2 + 8x + 2y + 7 = 0$ and $x^2 + y^2 + 2x - 16y + 25 = 0$ intersect at the point p. Find the coordinates of the point p.

10. The line $2x + y - 10 = 0$ is a common tangent to the circles $x^2 + y^2 - 2x + 4y + h = 0$ and $x^2 + y^2 - 14x - 2y + k = 0$.
Find the value of h and the value of k.

11. A circle C has centre (5, 3) and makes a chord of 4 units on the y-axis.
Find the equation of C.

12. A circle S has centre (2, 0) and makes a chord of 6 units on the y-axis.
Find the equation of S.

13. A circle of radius length 5 units has its centre in the first quadrant, touches the x-axis and intercepts a chord of length 6 units on the y-axis. Find the equation of this circle.

14. Find the equation of the circle with its centre in the first quadrant if it touches the y-axis at the point (0, 2) and makes a chord of length $4\sqrt{3}$ units on the x-axis.

15. L: $3x - 4y - 5 = 0$ is a line and C: $x^2 + y^2 - 4x + 12y + k = 0$ is a circle.
The line L contains a chord of length 10 units of the circle C.
Find: **(i)** the radius length of C **(ii)** the value of k.

16. The equation of a circle C is $x^2 + y^2 + 4x - 2y + k$. Write down its centre.
The midpoint of a chord of length $4\sqrt{2}$ is $(1, 4)$.
Find: **(i)** the distance from the centre of the circle to the chord.
(ii) the radius length of C and the value of k.

17. Find the equation of the circle with centre $(4, 1)$ and which makes an intercept of length 4 units on the line $3x - 4y + 2 = 0$.