CHAPTER 1

COORDINATE GEOMETRY OF THE CIRCLE

Equation of a Circle, Centre (0, 0) and Radius r

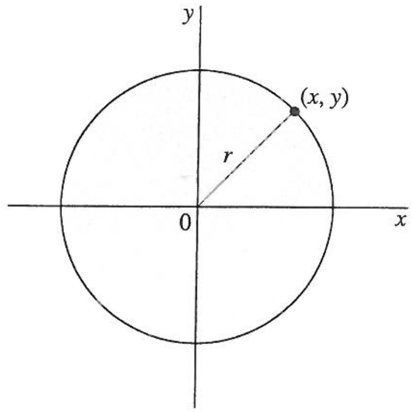
A circle is a set of points (a locus) which are equidistant from a fixed point called the '**centre**'.
The distance from the centre to any point on the circle is called the '**radius**'.
On the right is a circle with centre (0, 0), radius r and (x, y) any point on the circle.
Distance between (0, 0) and (x, y) equals the radius, r.

$\therefore \quad \sqrt{(x-0)^2+(y-0)^2}=r$ (distance formula)

$\sqrt{x^2+y^2}=r$

$x^2+y^2=r^2$ (square both sides)

Hence, $x^2+y^2=r^2$ is said to be the equation of the circle.



> Equation of a circle, centre (0, 0) and radius r, is
> $x^2+y^2=r^2$.

Two quantities are needed to find the equation of a circle:

> **1.** Centre **2.** Radius
> If the centre is (0, 0), the equation of the circle will be of the form $x^2+y^2=r^2$.

Example

Find the equations of the following circles, each of centre (0, 0):
(i) K_1, which has radius $\sqrt{13}$ **(ii)** K_2, which contains the point (4, –1).

Solution:

(i) Centre is (0, 0), therefore K_1 is of the form $x^2+y^2=r^2$.

$x^2+y^2=r^2$

$x^2+y^2=(\sqrt{13})^2$ (put in $\sqrt{13}$ for r)

$x^2+y^2=13$

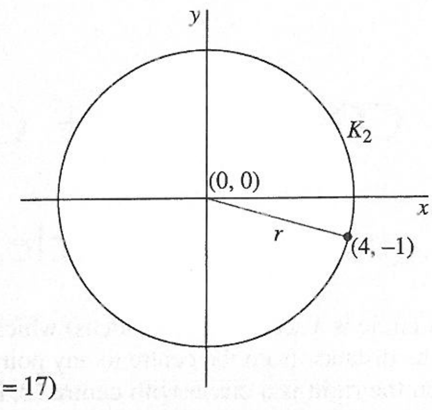
Thus, the equation of the circle K_1 is $x^2+y^2=13$.

(ii) Centre is (0, 0), therefore K_2 is of the form $x^2 + y^2 = r^2$.

$$x^2 + y^2 = r^2$$
$$(4)^2 + (-1)^2 = r^2 \qquad \text{(put in 4 for } x \text{ and } -1 \text{ for } y\text{)}$$
$$16 + 1 = r^2$$
$$17 = r^2$$

Thus, the equation of the circle K_2 is $x^2 + y^2 = 17$.

Alternatively, the radius is the distance from (0, 0) to (4, –1).
Using the distance formula, the radius
$= \sqrt{(4-0)^2 + (-1-0)^2} = \sqrt{16+1} = \sqrt{17}$.
Thus, the equation of the circle K_2 is $x^2 + y^2 = 17$. $\quad ((\sqrt{17})^2 = 17)$



Example

Find the centre and radius of each of the following circles:
(i) $x^2 + y^2 = 8$ **(ii)** $4x^2 + 4y^2 = 25$.

Solution:

(i) $x^2 + y^2 = 8$

In the form $x^2 + y^2 = r^2$

$\therefore$ the centre is (0, 0)

$$r^2 = 8$$
$$r = \sqrt{8} = 2\sqrt{2}$$

$\therefore$ the radius is $\sqrt{8}$ or $2\sqrt{2}$.

(ii) $4x^2 + 4y^2 = 25$

$$x^2 + y^2 = \frac{25}{4} \qquad \text{(divide each side by 4)}$$

In the form $x^2 + y^2 = r^2$

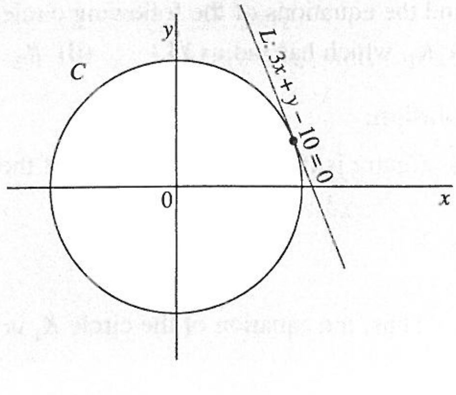
$\therefore$ the centre is (0, 0)

$$r^2 = \frac{25}{4}$$
$$r^2 = \sqrt{\frac{25}{4}} = \frac{\sqrt{25}}{\sqrt{4}} = \frac{5}{2}$$

$\therefore$ the radius is $\frac{5}{2}$.

Example

In the diagram, the line L: $3x + y - 10 = 0$ is a tangent to the circle C.
Find the equation of the circle C.



Solution:

As the centre of the circle C is (0, 0), its equation is of the form $x^2 + y^2 = r^2$.
We need to find the value of r.
The radius, r = perpendicular distance from the centre (0, 0) to the line $3x + y - 10 = 0$.
Using the formula for the perpendicular distance from a point to a line:

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

$$r = \frac{|3(0) + 1(0) - 10|}{\sqrt{3^2 + 1^2}} \qquad (a = 3, b = 1, c = -10, x_1 = 0, y_1 = 0)$$

$$r = \frac{|-10|}{\sqrt{10}} = \frac{10}{\sqrt{10}} = \sqrt{10}$$

Thus, the equation of the circle C is $x^2 + y^2 = 10$. $\quad ((\sqrt{10})^2 = 10)$

Exercise 1.1

In Q1 to Q14, find the equation of each of the following circles of centre (0, 0) and:

1. radius 2
2. radius 4
3. radius 5
4. radius $\sqrt{13}$
5. radius $\sqrt{2}$
6. radius $2\sqrt{3}$
7. radius $\frac{1}{2}$
8. radius $\frac{\sqrt{10}}{2}$
9. containing the point (3, 4)
10. containing the point (–5, 12)
11. containing the point (–1, –5)
12. containing the point (0, –3)
13. containing the point (–1, 1)
14. containing the point (2, –5)

Write down the radius length of each of the following circles:

15. $x^2 + y^2 = 16$
16. $x^2 + y^2 = 100$
17. $x^2 + y^2 = 1$
18. $x^2 + y^2 = 13$
19. $x^2 + y^2 = 5$
20. $x^2 + y^2 = 29$
21. $4x^2 + 4y^2 = 9$
22. $9x^2 + 9y^2 = 25$
23. $16x^2 + 16y^2 = 1$

24. Find the equation of the circle which has the line segment joining (3, –4) to (–3, 4) as diameter.

25. $a(6, 1)$ and $b(-6, -1)$ are two points. Find the equation of the circle with $[ab]$ as diameter.

26. (6, –3) is an extremity of a diameter of the circle $x^2 + y^2 = 45$. What are the coordinates of the other extremity of the same diameter?

27. What is the area of the circle $x^2 + y^2 = 40$? Leave your answer in terms of π.

Find the equation of each of the following circles, centre (0, 0) and having as a tangent the line:

28. $2x + y + 5 = 0$
29. $4x + y - 17 = 0$
30. $x + 3y + 10 = 0$
31. $5x - y - 26 = 0$
32. $x - y - 4 = 0$
33. $x + 2y - 10 = 0$

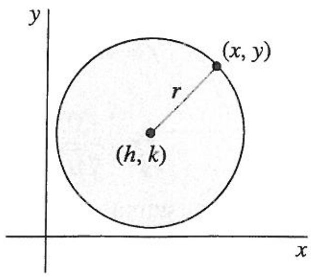
Equation of a Circle, Centre (h, k) and Radius r

On the right is a circle with centre (h, k) and radius r, and (x, y) is any point on the circle.
Distance between (h, k) and (x, y) equals the radius, r.

$\therefore\ \sqrt{(x-h)^2+(y-k)^2}=r$ (distance formula)

$(x-h)^2+(y-k)^2=r^2$ (square both sides)

Hence, $(x-h)^2+(y-k)^2=r^2$ is said to be the equation of the circle.



> The equation of a circle, centre (h, k) and radius r, is
> $(x-h)^2+(y-k)^2=r^2$.

Two quantities are needed to find the equation of a circle:

> **1.** Centre, (h, k) **2.** Radius, r
> Then use the formula $(x-h)^2+(y-k)^2=r^2$.

Note: If $(h, k)=(0, 0)$, the equation $(x-h)^2+(y-k)^2=r^2$ reduces to $x^2+y^2=r^2$.

Example

(i) Find the centre and radius of the circle $(x-2)^2+(y+5)^2=9$.

(ii) Find the equation of the circle, centre $(1, -4)$ and radius $\sqrt{13}$.

Solution:

(i) $(x-2)^2+(y+5)^2=9$

Compare exactly to:

$$\begin{array}{c}(x-h)^2+(y-k)^2=r^2\\ \downarrow \qquad\quad \downarrow \qquad \downarrow\\ (x-2)^2+(y+5)^2=9\end{array}$$

$\therefore\ h=2,\quad k=-5,\quad r=3$

Thus, centre $=(2, -5)$ and radius $=3$.

(ii) Centre $=(1, -4)$, radius $=\sqrt{13}$

$h=1,\quad k=-4,\quad r=\sqrt{13}$

$(x-h)^2+(y-k)^2=r^2$

$(x-1)^2+(y+4)^2=(\sqrt{13})^2$

$(x-1)^2+(y+4)^2=13$

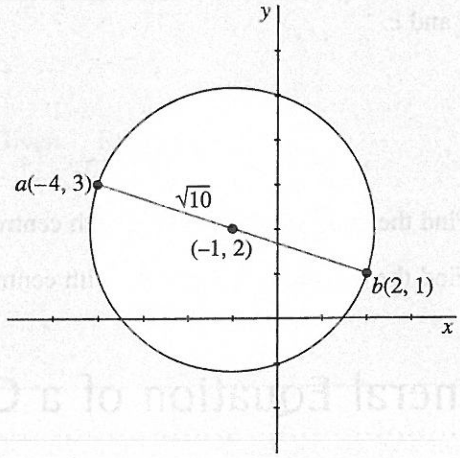
Example

Find the equation of the circle which has the line segment from $a(-4, 3)$ to $b(2, 1)$ as diameter.

Solution:

The **centre** and **radius** are needed.

The diagram on the right illustrates the situation.



Centre:

The centre is the midpoint of $[ab]$.

$$\text{Centre} = \left(\frac{-4+2}{2}, \frac{3+1}{2}\right) = \left(\frac{-2}{2}, \frac{4}{2}\right) = (-1, 2).$$

Radius:

The radius, r, is the distance from the centre $(-1, 2)$ to $(-4, 3)$ or $(2, 1)$.

Distance from $(-1, 2)$ to $(-4, 3)$

$$r = \sqrt{(-4+1)^2 + (3-2)^2} = \sqrt{(-3)^2 + (1)^2} = \sqrt{9+1} = \sqrt{10}$$

$$h = -1, \quad k = 2, \quad r = 10$$

Equation: $(x-h)^2 + (y-k)^2 = r^2$

$$(x+1)^2 + (y-2)^2 = (\sqrt{10})^2$$

$$(x+1)^2 + (y-2)^2 = 10$$

Exercise 1.2

Find the equation of each of the following circles, with given centre and radius:

1. Centre $(2, 3)$ and radius 4

2. Centre $(-2, -4)$ and radius 5

3. Centre $(-3, 2)$ and radius $\sqrt{10}$

4. Centre $(-3, 0)$ and radius $\sqrt{13}$

5. Centre $(0, 2)$ and radius $\frac{3}{2}$

6. Centre $(-1, 7)$ and radius $\frac{5}{2}$

Find the equation of the circle with:

7. centre $(1, 2)$ and containing the point $(2, 5)$

8. centre $(2, -1)$ and containing the point $(6, 4)$

9. centre $(-1, 3)$ and containing the point $(0, 5)$

10. centre $(-1, -3)$ and containing the point $(3, 0)$

11. centre $(-2, -5)$ and containing the point $(1, 1)$

12. centre $(4, -2)$ and containing the point $(0, 0)$

Find the coordinates of the centre and the length of the radius of each of the following circles:

13. $(x-2)^2 + (y-3)^2 = 25$

14. $(x+1)^2 + (y-2)^2 = 9$

15. $(x-5)^2 + (y+7)^2 = 1$

16. $(x+5)^2 + (y+7)^2 = 4$

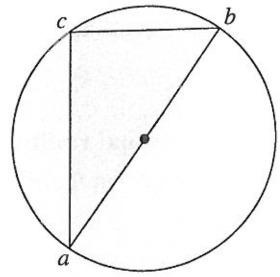
17. $x^2 + (y-5)^2 = 10$

18. $(x-4)^2 + y^2 = 13$

19. $a(5, 2)$ and $b(1, 4)$ are two points. Find the equation of the circle with $[ab]$ as diameter.

20. The end points of a diameter of a circle are $p(2, 4)$ and $q(-4, 0)$. Find the equation of the circle.

21. $a(-1, 5)$, $b(5, 13)$ and $c(-2, 12)$ are the vertices of triangle abc.
Show that the triangle is right angled at c.
Find the equation of the circle that passes through the coordinates a, b and c.



22. Find the equation of the circle with centre $(1, 3)$ and having the line $3x+4y+10=0$ as a tangent.

23. Find the equation of the circle with centre $(-6, 1)$ and having the line $x+y+1=0$ as a tangent.

General Equation of a Circle

The general equation of a circle is written as:

$$x^2+y^2+2gx+2fy+c=0$$

When the equation of a circle is given in this form, we use the following method to find its centre and radius.

1. Make sure every term is on the left-hand side and the coefficients of x^2 and y^2 are equal to 1.
2. Centre $=(-g, -f)=(-\frac{1}{2}$ coefficient of x, $-\frac{1}{2}$ coefficient of $y)$
3. Radius $=\sqrt{g^2+f^2-c}$ (provided $g^2+f^2-c>0$)

Notes: **1.** The equation is of the second degree (highest power is 2).
2. The coefficients of x^2 and y^2 are equal.
3. There is no xy term.

Example

Find the centre and radius of each of the circles:
(i) $x^2+y^2-4x+2y-11=0$ **(ii)** $x^2+y^2-8y+3=0$.

Solution:

(i) $x^2+y^2-4x+2y-11=0$
Centre $=(2, -1)$
Radius $=\sqrt{(2)^2+(-1)^2+11}$
$=\sqrt{4+1+11}=\sqrt{16}=4$

(ii) $x^2+y^2+0x-8y+3=0$ (put in $0x$)
Centre $=(0, 4)$
Radius $=\sqrt{(0)^2+(4)^2-3}$
$=\sqrt{0+16-3}=\sqrt{13}$

Example

The equation of a circle with radius 5 is $x^2+y^2-6x+4ky+20=0$, $\quad k \in \mathbf{Z}$.

(i) Find the centre of the circle and the radius length in terms of k.

(ii) Find the values of k.

Solution:

(i) $x^2+y^2-6x+4ky+20=0$

Centre $=(3,-2k)$

Radius $=\sqrt{(3)^2+(-2k)^2-20}$

$=\sqrt{9+4k^2-20}$

$=\sqrt{4k^2-11}$

(ii) Given: Radius $=5$

$\therefore \quad \sqrt{4k^2-11}=5$

$4k^2-11=25$

$4k^2=36$

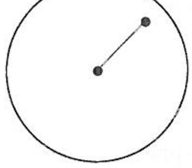
$k^2=9$

$k=\pm\sqrt{9}=\pm 3$

Points inside, on or outside a circle

Method 1:

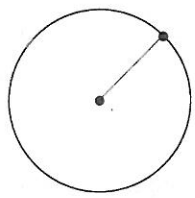
To find whether a point is inside, on or outside a circle, calculate the distance from the centre to the point and compare this distance with the radius. Three cases arise:



Inside

1. Distance from the centre, to the point is **less** than the radius

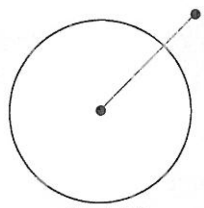
$\therefore$ point inside the circle.



On

2. Distance from the centre to the point is **equal** to the radius

$\therefore$ point on the circle.



Outside

3. Distance from the centre to the point is **greater** than the radius

$\therefore$ point outside the circle.

Method 2:

The equation of a circle can be of the form:

$$x^2+y^2=r^2$$
$$(x-h)^2+(y-k)^2=r^2$$
$$x^2+y^2+2gx+2fy+c=0$$

If the coordinates of a point satisfy the equation of a circle, then the point is **on** the circle. Otherwise, the point is either **inside** or **outside** the circle. By substituting the coordinates into the equation of the circle, one of the following situations can arise:

1. LHS < RHS: the point is **inside** the circle.
2. LHS = RHS: the point is **on** the circle.
3. LHS > RHS: the point is **outside** the circle.

Example

Determine whether the points (–3, –2), (5, –1), and (–2, 1) are inside, on or outside the circle $x^2+y^2-2x+8y-8=0$.

Solution: (using Method 2)

$$x^2+y^2-2x+8y-8=0$$

substitute (–3, –2): $(-3)^2+(-2)^2-2(-3)+8(-2)-8=9+4+6-16-8=-5<0$

$\therefore$ (–3, –2) is **inside** the circle.

substitute (5, –1): $(5)^2+(-1)^2-2(5)+8(-1)-8=25+1-10-8-8=0$

$\therefore$ (5, –1) is **on** the circle.

substitute (–2, 1): $(-2)^2+(1)^2-2(-2)+8(1)-8=4+1+4+8-8=9>0$

$\therefore$ (–2, 1) is **outside** the circle.

Exercise 1.3

Find the equation of each of the following circles with given centre and radius, writing your answers in the form $x^2+y^2+2gx+2fy+c=0$:

1. Centre (1, 2) and radius 3
2. Centre (–2, 3) radius 5
3. Centre (–3, –5) and radius $\sqrt{17}$
4. Centre (2, 0) and radius $\sqrt{10}$
5. Centre (0, –3) and radius $2\sqrt{2}$
6. Centre $(\frac{1}{2}, -\frac{1}{2})$ and radius $\sqrt{5}$
7. A circle with centre (–1, 3) passes through the point (1, –1). Find the equation of the circle.
8. A circle with centre (–3, –2) passes through the point (1, 1). Find the equation of the circle.

Find the centre and radius length of each of the following circles:

9. $x^2+y^2-6x-8y-11=0$
10. $x^2+y^2-4x-6y-3=0$
11. $x^2+y^2-2x+4y-4=0$
12. $x^2+y^2-10x+2y+6=0$
13. $x^2+y^2+8x-6y=0$
14. $x^2+y^2+2x-10y-10=0$
15. $x^2+y^2+6x-7=0$
16. $x^2+y^2=4y+4$
17. $2x^2+2y^2-2x-6y-13=0$
18. $9x^2+9y^2-6x+54y+46=0$
19. $(x-2)(x+4)+(y-1)(y-5)=3$
20. $(x-3)(x+3)+(y+2)(y+6)=0$

In each of the following, determine whether the given point is inside, on or outside the given circle:

21. (3, –2); $x^2+y^2=13$
22. (5, 3); $(x-3)^2+(y-2)^2=20$
23. (4, –1); $x^2+y^2+6x-4y-3=0$
24. (–1, 5); $x^2+y^2+4x-6y-25=0$
25. (4, 3); $x^2+y^2-4x+2y-15=0$
26. (–1, 4); $x^2+y^2+10x-6y+21=0$
27. The circle C has the equation $x^2+y^2+2x+2y-32=0$.
    The point (–4, k) lies on C. Find the two real values of k.

28. The circle S has the equation $(x-4)^2+(y-2)^2=13$.
The point $(p, 0)$ lies on S. Find the two real values of p.

29. The equation of a circle with radius length 4 is $x^2+y^2-6x+2y+k=0$, $k \in \mathbf{Z}$.
Find the value of k.

centre radius
$(3,-1)$ 4
$k = 3^2+(-1)^2-4$
$= 9+1-4=6$

30. The equation of a circle with radius length 6 is $x^2+y^2-2kx+4y-7=0$, $k \in \mathbf{Z}$.
(i) Find the centre of the circle and the radius length in terms of k.
(ii) Find the values of k.

31. The equation of a circle with radius length 5 is $x^2+y^2+2x-4ky+12=0$, $k \in \mathbf{Z}$.
Find the values of k.

32. $a(k, 1)$ and $b(-7, -k)$ are end points of a diameter of circle C.
If the centre of C is $(2, -5)$, find the value of k, and the radius length of C.

33. $a(-2, 0)$ and $b(6, 2)$ are points of a circle of centre $c(2k, k)$.
(i) Express in terms of k: **(a)** $|ac|^2$ **(b)** $|bc|^2$.
(ii) Find the value of k and the equation of the circle.

Proving that a Locus is a Circle

If the locus of a set of points is of the form $x^2+y^2+2gx+2fy+c=0$, then the locus is a circle and we can find its centre and radius.

Example

$p(1, 2)$, $q(1, -1)$ and $r(x, y)$ are points such that $2|qr|=|pr|$.
Show that $r(x, y)$ is on a circle. Find the centre and the length of the radius of the circle.

Solution:

$p(1, 2)$, $q(1, -1)$, $r(x, y)$

$$|qr|=\sqrt{(x-1)^2+(y+1)^2}=\sqrt{x^2-2x+1+y^2+2y+1}=\sqrt{x^2+y^2-2x+2y+2}$$

$$|pr|=\sqrt{(x-1)^2+(y-2)^2}=\sqrt{x^2-2x+1+y^2-4y+4}=\sqrt{x^2+y^2-2x-4y+5}$$

Given: $2|qr|=|pr|$

$$2\sqrt{x^2+y^2-2x+2y+2}=\sqrt{x^2+y^2-2x-4y+5}$$

$$(2\sqrt{x^2+y^2-2x+2y+2})^2=(\sqrt{x^2+y^2-2x-4y+5})^2 \qquad \text{(square both sides)}$$

$$4(x^2+y^2-2x+2y+2)=(x^2+y^2-2x-4y+5)$$

$$4x^2+4y^2-8x+8y+8=x^2+y^2-2x-4y+5$$

$$3x^2+3y^2-6x+12y+3=0$$

$$x^2+y^2-2x+4y+1=0 \qquad \text{(divide both sides by 3)}$$

This is the equation of a circle, as it is written in the form $x^2+y^2+2gx+2fy+c=0$.
$x^2+y^2-2x+4y+1=0$
Centre $=(1,-2)$ Radius $=\sqrt{(1)^2+(-2)^2-1}=\sqrt{1+4-1}=\sqrt{4}=2$

Exercise 1.4

1. $p(1, 5)$, $q(-5, -3)$ and $r(x, y)$ are three points such that $pr \perp rq$.
   (i) Find, in terms of x and y, the slope of: **(a)** pr **(b)** rq.
   (ii) Hence, or otherwise, find the equation of the circle that passes through p, q and r.

2. $p(1, 0)$, $q(4, 0)$ and $r(x, y)$ are points such that $2|qr| = |pr|$.
   Show that $r(x, y)$ is on a circle. Find the centre and the radius length of the circle.

3. $a(0, -1)$, $b(6, -1)$ and $c(x, y)$ are points such that $2|ac| = |bc|$.
   Show that the locus of c is a circle. Find the centre and radius length of the circle.

4. $a(2, -1)$, $b(10, 15)$ are two points. A point $c(x, y)$ moves such that $9|ac|^2 + 288 = |bc|^2$.
   Show that the locus of c is a circle. Find the centre and radius length of the circle.

Parametric Equations of a Circle

A circle may be defined by a pair of parametric equations. On our course we shall meet two types of parametric equation.

1. Trigonometric Parametric Equations

$$x = h \pm r\cos\theta, \qquad y = k \pm r\sin\theta$$
are the parametric equations of the circle
$$(x-h)^2 + (y-k)^2 = r^2.$$

We use the fact that $\cos^2\theta + \sin^2\theta = 1$.

Example

The parametric equations of a circle are $x = -2 + \sqrt{3}\cos\theta$, $y = 1 + \sqrt{3}\sin\theta$.
Find its Cartesian equation. Find its centre and radius length.

Solution:

$$x = -2 + \sqrt{3}\cos\theta$$
$$x + 2 = \sqrt{3}\cos\theta$$
$$\frac{x+2}{\sqrt{3}} = \cos\theta$$

$$y = 1 + \sqrt{3}\sin\theta$$
$$y - 1 = \sqrt{3}\sin\theta$$
$$\frac{y-1}{\sqrt{3}} = \sin\theta$$

Thus, $$\left(\frac{x+2}{\sqrt{3}}\right)^2 + \left(\frac{y-1}{\sqrt{3}}\right)^2 = \cos^2\theta + \sin^2\theta$$

$$\frac{(x+2)^2}{3} + \frac{(y-1)^2}{3} = 1 \qquad (\cos^2\theta + \sin^2\theta = 1)$$

$$(x+2)^2 + (y-1)^2 = 3 \qquad \text{(multiply both sides by 3)}$$

This is the Cartesian equation of the circle, as it is in the form $(x-h)^2 + (y-k)^2 = r^2$.
The centre is $(-2, 1)$ and the radius is $\sqrt{3}$.

2. Algebraic Parametric Equations

We can also represent the equation of a circle with x and y written as algebraic functions of the parameter t. For example:

$$x=\frac{6t}{1+t^2}, \quad y=\frac{3(1-t^2)}{1+t^2}$$

Example

The parametric equations of a circle are $x=\frac{1-t^2}{1+t^2}$, $y=\frac{2t}{1+t^2}$.

Find its Cartesian equation. Write your answer in the form $x^2+y^2=r^2$.

Solution:

$$\begin{aligned} x^2+y^2 &= \left(\frac{1-t^2}{1+t^2}\right)^2+\left(\frac{2t}{1+t^2}\right)^2 \\ &= \frac{1-2t^2+t^4}{1+2t^2+t^4}+\frac{4t^2}{1+2t^2+t^4} \\ &= \frac{1+2t^2+t^4}{1+2t^2+t^4} \qquad \text{(same base)} \\ &= 1 \end{aligned}$$

Thus, the Cartesian equation is $x^2+y^2=1$.

Exercise 1.5

Find the Cartesian equation of each of the following circles:

1. $x=2\cos\theta,\quad y=2\sin\theta$

2. $x=\sqrt{3}\cos\theta,\quad y=\sqrt{3}\sin\theta$

3. $x=1+\cos\theta,\quad y=2+\sin\theta$

4. $x=-3+\cos\theta,\quad y=4+\sin\theta$

5. $x=3+2\cos\theta,\quad y=5+2\sin\theta$

6. $x=-2+3\cos\theta,\quad y=-5+3\sin\theta$

7. $x=1+4\cos\theta,\quad y=-2+4\sin\theta$

8. $x=-5+5\cos\theta,\quad y=3+5\sin\theta$

9. $x=3+\sqrt{2}\cos\theta,\quad y=-5+\sqrt{2}\sin\theta$

10. $x=-2+\sqrt{8}\cos\theta,\quad y=1+\sqrt{8}\sin\theta$

11. $x=-3+\frac{1}{2}\cos\theta,\quad y=-1+\frac{1}{2}\sin\theta$

12. $x=3+\frac{\sqrt{3}}{2}\cos\theta,\quad y=-2+\frac{\sqrt{3}}{2}\sin\theta$

Find the Cartesian equation of each of the following circles, writing your answers in the form $x^2+y^2=k$. Find the radius in each case.

13. $x=t,\quad y=\pm\sqrt{9-t^2}$

14. $x=t-1,\quad y=\pm\sqrt{3+2t-t^2}$

15. $x=2t,\quad y=\pm\sqrt{16-4t^2}$

16. $x=\frac{2(1-t^2)}{1+t^2},\quad y=\frac{4t}{1+t^2}$

17. $x=\frac{t^2-4}{t^2+4},\quad y=\frac{4t}{t^2+4}$

18. $x=\frac{6t}{t^2+1},\quad y=\frac{3(t^2-1)}{t^2+1}$

19. The parametric equations of a circle are $x = a + 3a\cos\theta$, $y = -2a + 3a\sin\theta$, $a \in \mathbf{R}$. Find its Cartesian equation. Find, in terms of a, its centre and radius length.

20. The parametric equations of a circle are $x = \dfrac{a(1-t^2)}{1+t^2}$, $y = \dfrac{2at}{1+t^2}$, $a \in \mathbf{R}$.

Find its Cartesian equation and its radius length.

Sometimes we have to deal with algebraic parametric equations where the centre of the circle is not (0, 0). Consider the next example:

Example

Show that the parametric equations $x = \dfrac{2t}{1+t^2}$, $y = \dfrac{3+t^2}{1+t^2}$, $t \in \mathbf{R}$, represent a circle, and find its centre and its radius length.

Solution:

$$x = \frac{2t}{1+t^2} \qquad y = \frac{3+t^2}{1+t^2},$$

(Begin with y as it contains t^2, but not t)

$$y = \frac{3+t^2}{1+t^2}, \qquad x = \frac{2t}{1+t^2}$$

$$y + t^2y = 3 + t^2 \qquad x^2 = \frac{4t^2}{(1+t^2)^2} \quad \left(\text{square both sides}\right)$$

$$t^2y - t^2 = 3 - y \qquad x^2 = \frac{4\left(\frac{3-y}{y-1}\right)}{\left(1+\frac{3-y}{y-1}\right)^2} \quad \left(\text{put in } t^2 = \frac{3-y}{y-1}\right)$$

$$t^2(y-1) = 3 - y \qquad x^2 = \frac{4\left(\frac{3-y}{y-1}\right)}{\left(\frac{y-1+3-y}{y-1}\right)^2}$$

$$t^2 = \frac{3-y}{y-1} \qquad x^2 = \frac{4\left(\frac{3-y}{y-1}\right)}{\left(\frac{2}{y-1}\right)^2}$$

$$x^2 = \frac{4\left(\frac{3-y}{y-1}\right)}{\frac{4}{(y-1)^2}}$$

$$x^2 = 4\left(\frac{3-y}{y-1}\right) \cdot \frac{(y-1)^2}{4}$$

$$x^2 = (3-y)(y-1)$$
$$x^2 = 3y - 3 - y^2 + y$$
$$x^2 + y^2 - 4y + 3 = 0$$

This is the equation of a circle, as it is in the form $x^2 + y^2 + 2gx + 2fy + c = 0$.

Its centre is (0, 2) and its radius length $= \sqrt{0^2 + 2^2 - 3} = \sqrt{4-3} = \sqrt{1} = 1$.

Find the Cartesian equation of each of the following circles. Find its centre and radius length in each case:

21. $x = \dfrac{3+t^2}{1+t^2}$, $y = \dfrac{2t}{1+t^2}$

22. $x = \dfrac{1}{1+4t^2}$, $y = \dfrac{2t}{1+4t^2}$

Intersection of a Line and a Circle

To find the points where a line and a circle meet, the '**method of substitution**' between their equations is used.

The method involves the following three steps:

1. Get x or y on its own from the equation of the line.
   (Look carefully and select the variable which will make the working easier.)
2. Substitute for this same variable into the equation of the circle and solve the resultant quadratic equation.
3. Substitute **separately** the value(s) obtained in step 2 into the linear equation in step 1 to find the corresponding value(s) of the other variable.

Note: If there is only **one point of intersection** between a line and a circle, then the line is a **tangent** to the circle.

Example

The equation of a circle is $x^2 + y^2 + 4x - 2y - 5 = 0$.

The line $x - 2y - 1 = 0$ intersects the circle at the points p and q.

Find the coordinates of p and the coordinates of q.

Solution:

1. Get x or y on its own from the line:

$$x - 2y - 1 = 0$$
$$x = (2y + 1) \qquad [x \text{ on its own}]$$

2. Substitute $(2y+1)$ for x into the equation of the circle:

$$x^2+y^2+4x-2y-5=0$$

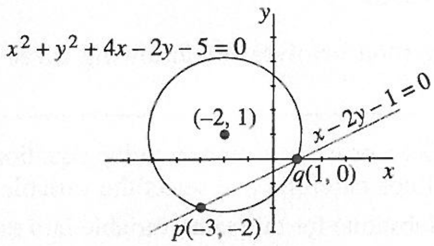
$$(2y+1)^2+y^2+4(2y+1)-2y-5=0 \qquad \text{[put in } (2y+1) \text{ for } x]$$
$$4y^2+4y+1+y^2+8y+4-2y-5=0$$
$$5y^2+10y=0$$
$$y^2+2y=0 \qquad \text{[divide both sides by 5]}$$
$$y(y+2)=0$$
$$\therefore \quad y=0 \quad \text{or} \quad y=-2$$

3. Substitute, separately, $y=0$ and $y=-2$ into the equation of the line in step 1 to find the x coordinates:

$y=0$	$y=-2$
$x=2y+1$	$x=2y+1$
$x=2(0)+1$	$x=2(-2)+1$
$x=1$	$x=-4+1=-3$
point is $(1, 0)$	point is $(-3, -2)$

Thus, the coordinates of the points of intersection are $p(-3, -2)$ and $q(1, 0)$.

The diagram on the right illustrates the situation.



Example

L: $3x-y+8=0$ is a line and C: $x^2+y^2-4x-8y+10=0$ is a circle.

Verify that L is a tangent to C and find the point of contact.

Solution:

As we need the point of contact, we use an algebraic approach.

1. Get x or y on its own from the line:

$$3x-y+8=0$$
$$-y=-3x-8$$
$$y=(3x+8) \qquad [y \text{ on its own}]$$

2. Substitute $(3x+8)$ for y into the equation of the circle:

$$x^2+y^2-4x-8y+10=0$$

$$x^2+(3x+8)^2-4x-8(3x+8)+10=0 \qquad \text{[put in } (3x+8) \text{ for } y]$$

$$x^2 + 9x^2 + 48x + 64 - 4x - 24x - 64 + 10 = 0$$
$$10x^2 + 20x + 10 = 0$$
$$x^2 + 2x + 1 = 0$$
$$(x + 1)(x + 1) = 0$$
$$\therefore \quad x = -1 \quad \text{or} \quad x = -1$$

3. Substitute $x = -1$ into the equation of the line in step 1 to find the y coordinate:

$x = -1$

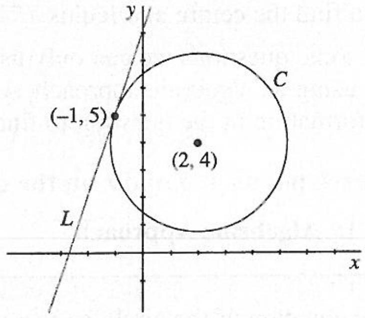
$y = 3x + 8$

$y = 3(-1) + 8$

$y = -3 + 8$

$y = 5$

point of contact $(-1, 5)$



Since there is only **one point of contact**, $(-1, 5)$, between L and C, the line L is a tangent to the circle C. The diagram on the right illustrates the situation.

Note: To show that a given line is a tangent to a circle, it is sufficient to show that the perpendicular distance from the centre of the circle to the line is equal to the radius. However, to find the point of contact an algebraic method (as shown above) is required.

Exercise 1.6

Find the coordinates of the point, or points, of intersection of the given line and circle. State whether or not the line is a tangent to the circle.

1. $x - 3y = 0$; $x^2 + y^2 = 10$

2. $x + 2y - 5 = 0$; $x^2 + y^2 = 10$

3. $x + 3y - 5 = 0$; $x^2 + y^2 = 5$

4. $4x - y - 17 = 0$; $x^2 + y^2 = 17$

5. $x - y - 1 = 0$; $x^2 + y^2 - 2x - 2y + 1 = 0$

6. $x - 2y - 1 = 0$; $x^2 + y^2 + 2x - 8y - 8 = 0$

7. $2x - y + 8 = 0$; $x^2 + y^2 + 4x + 2y = 0$

8. $x - 3y + 5 = 0$; $x^2 + y^2 - 6x - 2y - 15 = 0$

9. $x + 2y - 7 = 0$; $x^2 + y^2 - 2x + 4y - 15 = 0$

10. $x - 4y - 6 = 0$; $x^2 + y^2 + 6x - 4y - 4 = 0$

11. $5x - 3y - 17 = 0$; $x^2 + y^2 = 17$

12. $3x + 2y - 20 = 0$; $x^2 + y^2 - 6x + 2y - 3 = 0$

13. The line $x - 2y - 3 = 0$ intersects the circle $(x - 2)^2 + (y + 3)^2 = 25$ at p and q. Calculate $|pq|$.

14. The equation of a circle is $(x - 2)^2 + (y - 1)^2 = 10$.
The line $x - 3y + 1 = 0$ intersects the circle at points a and b.
(i) Find the coordinates of a and the coordinates of b.
(ii) Investigate whether $[ab]$ is a diameter of the circle.

Finding the Equation of a Circle

If the centre and radius are given, or can be found, then using the formula

$$(x-h)^2+(y-k)^2=r^2$$

is the preferred method for finding the equation of a circle.
However, for many questions it is difficult to find the centre and radius.
In these questions we have to use an algebraic approach, or rely on our knowledge of the geometry of a circle to find the centre and radius.

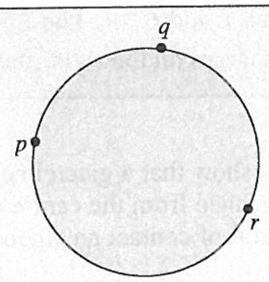
Note: In some questions we can only use an algebraic approach.
In using an algebraic approach, we let the circle be $x^2+y^2+2gx+2fy+c=0$ and use the information in the question to find g, f and c.

Given three points p, q and r on the circle

Method 1: Algebraic Approach

Let the equation of the circle be $x^2+y^2+2gx+2fy+c=0$.

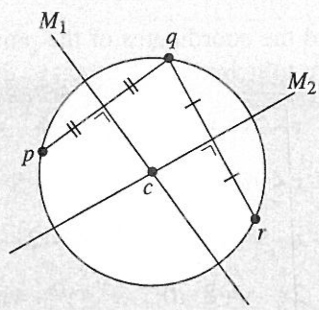
1. Substitute each point into this equation.
2. This gives three equations in three unknowns: g, f and c.
3. Solve these equations for g, f and c.
4. Put these values back into the equation.



Method 2: Geometric Approach

1. Find the equations of the perpendicular bisectors M_1 and M_2 of the chords $[pq]$ and $[qr]$, respectively.
(The perpendicular bisector of a chord passes through the centre.)
2. The centre of the circle is $c=M_1 \cap M_2$.
(Solve the equations of M_1 and M_2 simultaneously.)
3. The radius is the distance from c to p, q or r.
4. Use the formula: $(x-h)^2+(y-k)^2=r^2$.

Note: There is only one circle that contains the points p, q and r.



Example

Points $p(-4, 2)$, $q(-2, 6)$ and $r(4, 8)$ are on a circle S. Find the equation of S.

Solution:

Method 1: Algebraic approach

We are given three points on the circumference of the circle.

Let the equation of the circle be $x^2+y^2+2gx+2fy+c=0$.

We need the values of g, f and c.

$(-4, 2)$ on the circle: $(-4)^2+(2)^2+2g(-4)+2f(2)+c=0 \Rightarrow 8g-4f-c=20$ ①

$(-2, 6)$ on the circle: $(-2)^2+(6)^2+2g(-2)+2f(6)+c=0 \Rightarrow 4g-12f-c=40$ ②

$(4, 8)$ on the circle: $(4)^2+(8)^2+2g(4)+2f(8)+c=0 \Rightarrow 8g+16f+c=-80$ ③

We now solve between the simultaneous equations ①, ② and ③.

Eliminate c from two different pairs of equations:

$$\begin{aligned} 8g-4f-c &= 20 && ① \\ 8g+16f+c &= -80 && ③ \\ \hline 16g+12f &= -60 && \text{(add)} \\ 4g+3f &= -15 && ④ \end{aligned}$$

$$\begin{aligned} 4g-12f-c &= 40 && ② \\ 8g+16f+c &= -80 && ③ \\ \hline 12g+4f &= -40 && \text{(add)} \\ 3g+f &= -10 && ⑤ \end{aligned}$$

Now solve between ④ and ⑤ to find the values of g and f.

$$\begin{aligned} 4g+3f &= -15 && ④ \\ -9g-3f &= 30 && ⑤ \times -3 \\ \hline -5g &= 15 \\ 5g &= -15 \\ g &= -3 \end{aligned}$$

$$\begin{aligned} 3g+f &= -10 && ⑤ \\ 3(-3)+f &= -10 \\ -9+f &= -10 \\ f &= -1 \end{aligned}$$

Put $g=-3$ into ④ or ⑤.

Put $g=-3$ and $f=-1$ into ①, ② or ③ to find the value of c.

$$\begin{aligned} 8g-4f-c &= 20 \quad ① \\ 8(-3)-4(-1)-c &= 20 \\ -24+4-c &= 20 \\ -20-c &= 20 \\ -c &= 40 \\ c &= -40 \end{aligned}$$

The equation of the circle S is:

$$x^2+y^2+2gx+2fy+c=0$$

$$x^2+y^2+2(-3)x+2(-1)y-40=0$$

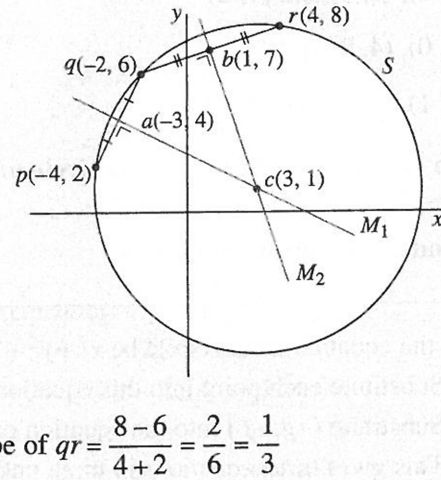
(put in $g=-3, f=-1$ and $c=-40$)

$$x^2+y^2-6x-2y-40=0$$

Method 2: Geometric approach

The diagram on the right represents the situation.

We find the equation of M_1, the perpendicular bisector of $[pq]$, and then the equation of M_2, the perpendicular bisector of $[qr]$. The centre, c, is the point of intersection of M_1 and M_2. The radius is $|pc|$, $|qc|$ or $|rc|$.



Slope of $pq = \dfrac{6-2}{-2+4} = \dfrac{4}{2} = 2$

$\therefore$ Slope of $M_1 = -\dfrac{1}{2}$

Slope of $qr = \dfrac{8-6}{4+2} = \dfrac{2}{6} = \dfrac{1}{3}$

$\therefore$ Slope of $M_2 = -3$

Midpoint of $[pq] = a\left(\frac{-4-2}{2}, \frac{2+6}{2}\right)$

$= a(-3, 4)$

Equation of M_1 [slope $= -\frac{1}{2}$, point $= (-3, 4)$]:

$$(y - y_1) = m(x - x_1)$$
$$(y - 4) = -\tfrac{1}{2}(x + 3)$$
$$2y - 8 = -x - 3$$
$$M_1\colon x + 2y - 5 = 0$$

Midpoint of $[qr] = b\left(\frac{-2+4}{2}, \frac{6+8}{2}\right)$

$= b(1, 7)$

Equation of M_2 [slope $= -3$, point $= (1, 7)$]:

$$(y - y_1) = m(x - x_1)$$
$$(y - 7) = -3(x - 1)$$
$$y - 7 = -3x + 3$$
$$M_2\colon 3x + y - 10 = 0$$

Solving the simultaneous equations M_1 and M_2 gives the centre of the circle $c(3, 1)$.

The radius of S is $|pc|$ or $|qc|$ or $|rc|$.

$p(-4, 2)$, $c(3, 1)$.

$$|pc| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(3+4)^2 + (1-2)^2} = \sqrt{(7)^2 + (-1)^2} = \sqrt{49 + 1} = \sqrt{50}$$

Thus, the centre of S is $(3, 1)$ and the radius length is $\sqrt{50}$.

Equation of S:
$$(x - h)^2 + (y - k)^2 = r^2$$
$$(x - 3)^2 + (y - 1)^2 = (\sqrt{50})^2$$
$$(x - 3)^2 + (y - 1)^2 = 50$$

or

$$x^2 + y^2 - 6x - 2y - 40 = 0$$

Exercise 1.7

Find the equation of the circle which contains the points:

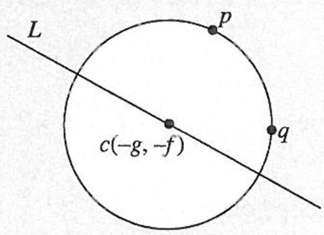
1. $(2, 2)$, $(6, 4)$ and $(4, 8)$
2. $(-3, -4)$, $(-5, 2)$ and $(1, 8)$
3. $(0, 0)$, $(4, 0)$ and $(6, -2)$
4. $(10, -2)$, $(-2, 4)$ and $(2, -2)$
5. $(4, 1)$, $(-2, 1)$ and $(2, 3)$
6. $(-2, -1)$, $(0, -5)$ and $(1, -2)$

Given two points p and q on the circle and the equation of a line, L, containing the centre $c(-g, -f)$

Method 1: Algebraic Approach

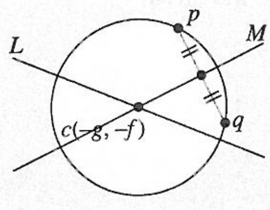
Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$.

1. Substitute each point into this equation.
2. Substitute $(-g, -f)$ into the equation of the given line.
3. This gives three equations in three unknowns: g, f and c.
4. Solve these equations for g, f and c.
5. Put these values back into the equation.



Method 2: Geometric Approach

1. Find the equation of M, the perpendicular bisector of $[pq]$.
2. Solve the simultaneous equations L and M to find the centre.
3. Find the radius r, the distance from the centre c to p or q.
4. Use the formula $(x-h)^2+(y-k)^2=r^2$.



Example

Find the equation of the circle that contains the points (–4, 1), and (0, 3) and whose centre lies on the line $x-2y+11=0$.

Solution:

Method 1: Algebraic approach

We are given two points on the circumference and the equation of a line containing the centre.

Let the equation of the circle be $x^2+y^2+2gx+2fy+c=0$.

(–4, 1) on the circle: $(-4)^2+(1)^2+2g(-4)+2f(1)+c=0 \Rightarrow 8g-2f-c=17$ ①

(0, 3) on the circle: $(0)^2+(3)^2+2g(0)+2f(3)+c=0 \Rightarrow 6f+c=-9$ ②

The centre $(-g, -f)$ is on the line $x-2y+11=0$.

Thus, $(-g)-2(-f)+11=0 \Rightarrow g-2f=11$ ③

Eliminate c from ① and ②:

$8g-2f-c=17$ ①

$6f+c=-9$ ②

$8g+4f=8$ (add)

$2g+f=2$ ④

Now solve between ③ and ④ to find the values of g and f:

$g-2f=11$ ③

$4g+2f=4$ ④ × 2

$5g=15$

$g=3$

Put $g=3$ into ③ or ④.

$2g+f=2$ ④

$2(3)+f=2$

$6+f=2$

$f=-4$

Put $g=3$ and $f=-4$ into ① or ② to find the value of c.

$6f+c=-9$ ②

$6(-4)+c=-9$

$-24+c=-9$

$c=15$

The equation of the circle is:

$x^2+y^2+2gx+2fy+c=0$

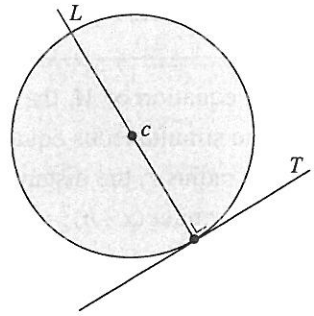
$x^2+y^2+2(3)x+2(-4)y+15=0$

(put in $g=3, f=-4$ and $c=15$)

$x^2+y^2+6x-8y+15=0$

Note: Using Method 2, the geometric approach gives the same result.

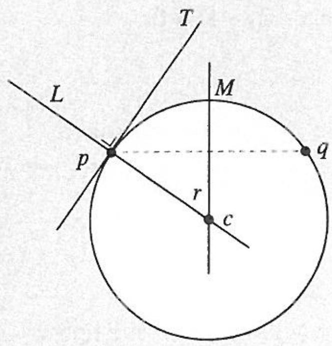
Note: A line perpendicular to a tangent at the point of tangency passes through (contains) the centre of the circle.



Given two points, p and q, on the circle and the equation of the tangent at one of these points

Method:

1. Find the equation of L, the line perpendicular to the tangent T passing through the given point of contact. This line will contain the centre c.
2. Now we have two points on the circumference of the circle and the equation of a line that contains the centre of the circle.
3. Use an algebraic approach (as in the previous example) or use a geometric approach.



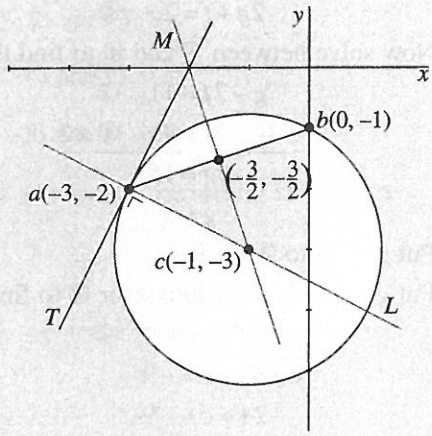
Example

Find the equation of the circle which passes through the points $a(-3, -2)$, and $b(0, -1)$ and where the line $2x - y + 4 = 0$ is a tangent at the point $a(-3, -2)$.

Solution:

Method 2: Geometric approach

The diagram on the right represents the situation. Find the equation of L, the line that is perpendicular to the tangent at the point of contact $a(-3, -2)$. This line contains the centre. Find the equation of M, the perpendicular bisector of $[ab]$. The centre c is the point of intersection of L and M. The radius is $|ac|$ or $|bc|$.



The slope of the tangent

$2x - y + 4 = 0$ is 2

$\therefore$ the slope of L is $-\frac{1}{2}$.

Slope of $ab = \dfrac{-1+2}{0+3} = \dfrac{1}{3}$

$\therefore$ slope of $M = -3$.

Midpoint of $[ab] = \left(\dfrac{-3+0}{2}, \dfrac{-2-1}{2}\right) = (-\frac{3}{2}, -\frac{3}{2})$