For a point of inflection:

$$\frac{d^2y}{dx^2} = 0$$

$$\therefore \quad 6x - 6 = 0$$

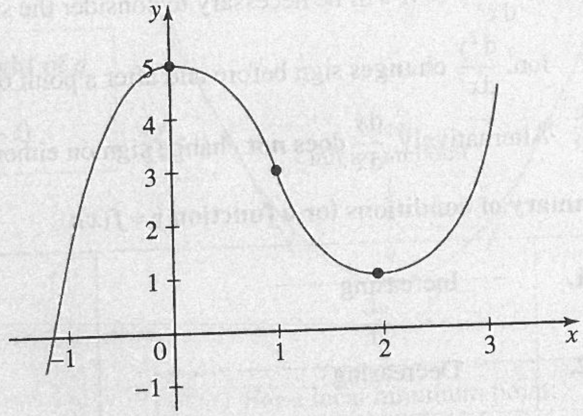
$$6x = 6$$

$$x = 1$$

$$x = 1; \quad y = (1)^3 - 3(1)^2 + 5 = 3$$

$\therefore$ point of inflection is $(1, 3)$

Check: $\dfrac{d^3y}{dx^3} = 6 \neq 0$.



Example

Let $f(x) = xe^{-ax}$, $x \in \mathbf{R}$, a constant and $a > 0$.

Show that $f(x)$ has a local maximum and express the coordinates of this local maximum point in terms of a.

Find, in terms of a, the coordinates of the point at which the second derivative of $f(x)$ is zero.

Solution:

For a maximuim: **1.** $f'(x) = 0$ and **2.** $f''(x) < 0$.

$$f(x) = xe^{-ax}$$

$$f'(x) = x(e^{-ax})(-a) + e^{-ax}(1)$$

$$= -axe^{-ax} + e^{-ax}$$

$$= e^{-ax}(1 - ax)$$

$$f''(x) = e^{-ax}(-a) + (1 - ax)(e^{-ax})(-a)$$

$$= -ae^{-ax} - ae^{-ax} + a^2xe^{-ax}$$

$$= ae^{-ax}(-1 - 1 + ax)$$

$$= ae^{-ax}(ax - 2)$$

$$f'(x) = 0$$

$$e^{-ax}(1 - ax) = 0$$

$$\therefore \quad 1 - ax = 0$$

$$ax = 1$$

$$x = \frac{1}{a}$$

Note: $e^{-ax} \neq 0$ for any value of x.

$$f''\left(\frac{1}{a}\right) = ae^{-a(1/a)}\left[a\left(\frac{1}{a}\right) - 2\right]$$

$$= ae^{-1}(1 - 2)$$

$$= -ae^{-1}$$

$$= -\frac{a}{e} < 0 \qquad (\text{as } a > 0)$$

$\therefore$ local maximum occurs at $x = \dfrac{1}{a}$

$$f(x) = xe^{-ax}$$

$$f\left(\frac{1}{a}\right) = \frac{1}{a}e^{-a(1/a)} = \frac{1}{a}e^{-1} = \frac{1}{a} \cdot \frac{1}{e} = \frac{1}{ae}$$

Thus, the coordinates of the local maximum point are $\left(\dfrac{1}{a}, \dfrac{1}{ae}\right)$.

$$f''(x) = 0$$
$$ae^{-ax}(ax - 2) = 0$$
$$ax - 2 = 0$$
$$ax = 2$$
$$x = \frac{2}{a}$$

Note: $ae^{-ax} \neq 0$ for any value of x.

$$f(x) = xe^{-ax}$$
$$f\left(\frac{2}{a}\right) = \frac{2}{a}\, e^{-a(2/a)}$$
$$= \frac{2}{a}\, e^{-2}$$
$$= \frac{2}{a} \cdot \frac{1}{e^2}$$
$$= \frac{2}{ae^2}$$

Thus, the coordinates of the point at which $f''(x) = 0$ are $\left(\frac{2}{a}, \frac{2}{ae^2}\right)$.

Exercise 13.3

Find the coordinates of the turning point of each of the following functions and determine if each turning point is a local maximum or local minimum:

1. $y = x^2 - 2x + 5$ **2.** $y = 3x^2 + 6x - 5$ **3.** $y = 1 - 12x - 2x^2$

Find the coordinates of the local maximum point, the local minimum point and the point of inflection of each of the following functions. Draw a rough graph of the function in each case:

4. $y = x^3 - 6x^2 + 9x - 5$ **5.** $y = 12x - x^3$ **6.** $y = x^3 - 9x^2 + 15x + 10$ **7.** $y = 2 - 3x^2 - x^3$

8. Let $f(x) = x + \frac{1}{x}$, $x \neq 0$. Find the coordinates of the local maximum and the local minimum of $f(x)$. Verify that $f(x)$ has no points of inflection.

9. If $f(x) = x^4 - 4x^3$, find the coordinates of any points of inflection.

10. $f(x) = x^4 - 2x^2$. Verify that $f(x)$ has one local maximum and two local minimum points, and calculate the coordinates of these points.
Find the coordinates of the two points of inflection of $f(x)$.

11. Let $f(x) = \frac{4x - 3}{x^2 + 1}$.
Calculate the coordinates of the local maximum point and the local minimum point of $f(x)$.

Find the coordinates of any turning points of each of the following and determine whether they are local maximum points or local minimum points:

12. $y = x \ln x - 2x$, $x > 0$ **13.** $y = \frac{\ln x}{x}$, $x > 0$ **14.** $y = e^{x^2}$

15. $y = xe^x$ **16.** $y = x^2e^{-x}$ **17.** $y = (1 - \ln x)^2$, $x > 0$.

18. Let $f(x) = xe^{-x}$. Find (i) $f'(x)$ and (ii) $f''(x)$.
Find the coordinates of the turning point and determine if it is a maximum or a minimum.
Find the coordinates of the point of inflection.

19. Let $x + y = 13$, where $x, y > 0$.
If $A = 2x + 3y + xy$, write A as a quadratic in x.
Calculate the maximum value of A.

20. Let $x + y = 12$, where $x, y > 0$.
If $A = x^2 + y^2$, calculate the minimum value of A.

21. Given that the curve $y = ax^2 + 12x + 1$ has a turning point at $x = 2$, calculate the value of a.
Is the point a maximum or a minimum?

22. The curve $y = px^2 + qx + r$ has a maximum turning point at $(2, 18)$.
If $(0, 10)$ is a point on the curve, find the value of p, q and r.

23. The curve $y = e^x(px^2 + q)$ has a local minimum point at $(1, -4e)$.
Find the value of p and the value of q.

24. Given that $y = e^{2x}\cos 2x$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

Verify that $e^{2x}\cos 2x$ has a maximum value at $x = \dfrac{\pi}{8}$ and write down this maximum value.

25. $y = e^{2x} - 2e^x$ have one turning point. Find its coordinates.
Determine if it is a local maximum or a local minimum point.

26. Let $f(x) = e^{2x} - ae^x$, $x \in \mathbf{R}$ and a constant, $a > 0$.
Show that $f(x)$ has a local minimum at a point $(b, f(b))$, specifying the value of b in terms of a.

27. Let $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$. Verify $\dfrac{d^3y}{dx^3} \neq 0$.

If $b^2 = 3ac$, show that $f(x)$ has only one turning point.

28. Let $f(x) = 2x^3 - kx^2 + \dfrac{10k^3}{27}$, $x \in \mathbf{R}$ and $k > 0$.

Find the coordinates of the local minimum and the local maximum points, in terms of k.

Asymptotes

An '**asymptote**' is a straight line that a curve approaches but never meets.

On our course we will meet two types of asymptote:

1. Vertical asymptote **2. Horizontal asymptote**

A **rational function** is a function of the form $f(x) = \dfrac{g(x)}{h(x)}$.

The rational functions on our course are ones of the form:

1. $f(x) = \dfrac{a}{x+b}$ **2.** $f(x) = \dfrac{x}{x+b}$

Properties of these rational functions:

> **1.** They have no turning points or points of inflection.
> **2.** They are always increasing or decreasing.
> **3.** Vertical asymptote: Bottom = 0, i.e. $x + b = 0$ or $x = -b$.
> **4.** Horizontal asymptote: $y = \lim_{x \to \infty} f(x)$.

Example

Let $f(x) = \dfrac{x}{x-3}$, $x \neq 3$ and $x \in \mathbf{R}$.

(i) Show that $f(x)$ has no turning points and that it is decreasing for all $x \neq 3$, in its domain.
(ii) Show that the curve $f(x)$ has no points of inflection.
(iii) Find the equations of the asymptotes of the curve $f(x)$.
(iv) Draw a sketch of the curve $f(x)$.
(v) Find how x_1 and x_2 are related if the tangents at $(x_1, f(x_1))$ and $(x_2, f(x_2))$ are parallel and $x_1 \neq x_2$.

Solution:

$$f(x) = \frac{x}{x-3}$$

(i)
$$f'(x) = \frac{(x-3)(1) - (x)(1)}{(x-3)^2} \qquad \text{(quotient rule)}$$
$$= \frac{x-3-x}{(x-3)^2}$$
$$= \frac{-3}{(x-3)^2}$$

$$\frac{-3}{(x-3)^2} < 0 \quad \text{for all } x \neq 3$$

(as top is always negative and bottom is always positive).
Thus, the curve has no stationary points and is decreasing for all $x \neq 3$.

(ii)
$$f'(x) = \frac{-3}{(x-3)^2} = -3(x-2)^{-2}$$
$$f''(x) = (-2)(-3)(x-3)^{-3}(1) \qquad \text{(chain rule)}$$
$$= 6(x-3)^{-3}$$
$$= \frac{6}{(x-3)^3}$$
$$f''(x) = 0 \qquad \text{(for a point of inflection)}$$
$$\Rightarrow \quad \frac{6}{(x-3)^3} = 0$$
$$\Rightarrow \quad 6 = 0 \qquad \text{(not true)}$$

Thus, $f''(x) \neq 0$

$\therefore$ No points of inflexion.

(iii) $f(x) = \dfrac{x}{x-3}$

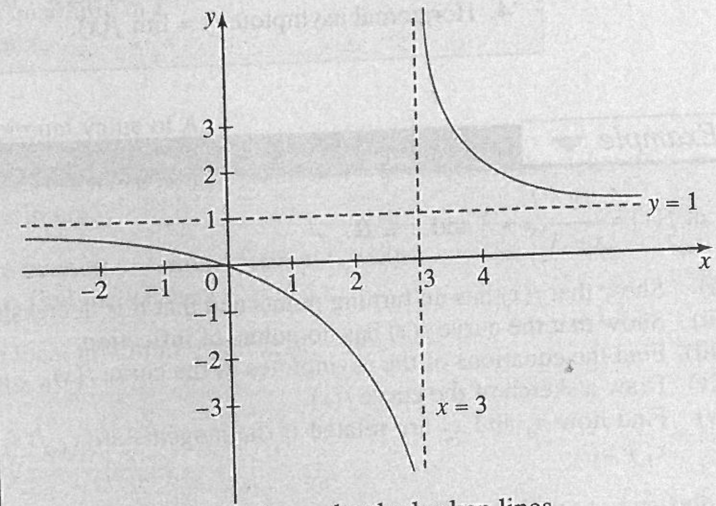
Vertical asymptote:

$$\text{Bottom} = 0$$
$$x - 3 = 0$$
$$x = 3$$

Horizontal asymptote:

$$y = \lim_{x \to \infty} f(x)$$
$$y = \lim_{x \to \infty} \frac{x}{x-3}$$
$$y = \lim_{x \to \infty} \frac{1}{1 - \frac{3}{x}}$$
$$y = \frac{1}{1-0}$$
$$y = 1$$

(iv) For the graph, $y = f(x)$.
When $x = 0$, $y = 0$,
thus, the point $(0, 0)$ is on the curve.
Sketch:



The asymptotes are shown by the broken lines.

(v)

$$\frac{dy}{dx} = \frac{-3}{(x-3)^2}$$
$$\left.\frac{dy}{dx}\right|_{x=x_1} = \frac{-3}{(x_1-3)^2}$$
$$\left.\frac{dy}{dx}\right|_{x=x_2} = \frac{-3}{(x_2-3)^2}$$

Parallel tangents

$$\frac{-3}{(x_1-3)^2} = \frac{-3}{(x_2-3)^2}$$
$$(x_1-3)^2 = (x_2-3)^2$$
$$(x_1-3) = \pm(x_2-3)$$
$$x_1 - 3 = x_2 - 3 \quad \text{or} \quad x_1 - 3 = -x_2 + 3$$
$$x_1 = x_2 \quad \text{or} \quad x_1 + x_2 = 6$$
$$\text{Thus, } x_1 + x_2 = 6 \quad [\text{as } x_1 \neq x_2]$$

Exercise 13.4

In each case, find the equations of the asymptotes of the graph of $f(x)$:

1. $f(x) = \dfrac{x}{x+2}$ **2.** $f(x) = \dfrac{4}{x-5}$ **3.** $f(x) = \dfrac{3}{x}$ **4.** $f(x) = \dfrac{x}{x-3}$

5. $f(x) = \dfrac{x}{x+1}$, where $x \in \mathbf{R}, x \neq -1$.

(i) Find the horizontal and vertical asymptotes of $y = f(x)$.
(ii) Show that $y = f(x)$ has no stationary points, and that it is increasing for all $x \neq -1$.
(iii) Draw a rough sketch of the curve $y = f(x)$.

6. $f(x) = \dfrac{2}{x-3}$, where $x \in \mathbf{R}, x \neq 3$.

(i) Show that $f(x)$ has no turning points and that it is decreasing for all $x \neq 3$, in its domain.
(ii) Show that the curve $f(x)$ has no points of inflection.
(iii) Find the equations of the asymptotes of the curve $f(x)$.
(iv) Draw a sketch of the curve $f(x)$.
(v) Find how x_1 and x_2 are related if the tangents at $(x_1, f(x_1))$ and $(x_2, f(x_2))$ are parallel and $x_1 \neq x_2$.

7. $f(x) = \dfrac{4}{x+2}$, where $x \in \mathbf{R}, x \neq -2$.

(i) Find the equations of the asymptotes of $f(x)$.
(ii) Draw a sketch of the curve of $f(x)$.
(iii) Show that the curve of $f(x)$ is always decreasing and has no points of inflection.
(iv) $x + y = 2$ is a tangent to the curve at the point $(0, 2)$. Find the point of tangency of the other tangent parallel to $x + y + 2$.

8. $f(x) = \dfrac{2}{x-2}$, where $x \in \mathbf{R}, x \neq 2$.

(i) Find the equations of the asymptotes of the graph of $f(x)$.
(ii) Prove that the graph of $f(x)$ has no turning points or points of inflection.
(iii) If the tangents to the curve at $x = x_1$ and $x = x_2$ are parallel and if $x_1 \neq x_2$, show that $x_1 + x_2 - 4 = 0$.

9. $f(x) = \dfrac{x}{x+4}$, $\quad x \in \mathbf{R}$ and $x \neq -4$.

(i) Find the equations of the asymptotes of the graph of $f(x)$.
(ii) Prove that the graph of $f(x)$ has no turning points or points of inflection.
(iii) Find the range of values of x for which $f'(x) \leqslant 1$, where $f'(x)$ is the derivative of $f(x)$.

Rates of Change I

Displacement (position), Velocity and Acceleration

The derivative $\dfrac{dy}{dx}$ is called the '**rate of change of *y* with respect to *x***'.

It shows how changes in y are related to changes in x.

If $\dfrac{dy}{dx} = 5$, then y is increasing 5 times as fast as x increases.

If $\dfrac{dy}{dx} = -3$, then y decreases 3 times as fast as x increases.

In mechanics, for example, letters other than x and y are used.

If s denotes the displacement (position) of a particle from a fixed point, at time t, then:

1. Velocity $= v = \dfrac{dy}{dx}$,
the rate of change of position with respect to time.

2. Acceleration $= a = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2}$,
the rate of change of velocity with respect to time.

Example

A particle moves along a straight line such that, after t seconds, the distance moved, s metres, is given by $s = t^3 - 9t^2 + 15t - 3$. Find:

(i) the velocity and acceleration of the particle, in terms of t
(ii) the values of t when its velocity is zero
(iii) the acceleration after $3\frac{1}{2}$ seconds
(iv) the time at which the acceleration is 6 m/s², and the velocity at this time.

Solution:

(i)
$$s = t^3 - 9t^2 + 15t - 3$$
$$v = \frac{ds}{dt} = 3t^2 - 18t + 15 \qquad \text{(velocity at any time } t\text{)}$$
$$a = \frac{d^2s}{dt^2} = 6t - 18 \qquad \text{(acceleration at any time } t\text{)}$$

(ii) Velocity = 0
$$\therefore \quad \frac{ds}{dt} = 0$$
$$3t^2 - 18t + 15 = 0$$
$$t^2 - 6t + 5 = 0$$
$$(t-1)(t-5) = 0$$
$$t = 1 \quad \text{or} \quad t = 5$$
Thus, the particle is stopped after 1 second and again after 5 seconds.

(iii)
$$\frac{d^2s}{dt^2} = 6t - 18$$
$$\therefore \quad \left.\frac{d^2s}{dt^2}\right|_{t=3\frac{1}{2}} = 6(3\tfrac{1}{2}) - 18$$
$$= 21 - 18$$
$$= 3 \text{ m/s}^2$$
Thus, after $3\frac{1}{2}$ seconds the acceleration is 3 m/s².

(iv) acceleration = 6 m/s²
$$\therefore \quad \frac{d^2s}{dt^2} = 6$$
$$\therefore \quad 6t - 18 = 6$$
$$6t = 24$$
$$t = 4$$
After 4 seconds the acceleration is 6 m/s².

Velocity after 4 seconds
$$\left.\frac{ds}{dt}\right|_{t=4} = 3(4)^2 - 18(4) + 15$$
$$= 48 - 72 + 15$$
$$= -9 \text{ m/s}$$
After 4 seconds the velocity is −9 m/s. The negative value means after 4 seconds it is going in the opposite direction to when it started.

Exercise 13.5

1. If $s = t^3 - 2t^2$, evaluate $\frac{ds}{dt}$ at $t = 3$.

2. If $\theta = 3t^2 - \frac{1}{3}t^3$, evaluate $\frac{d\theta}{dt}$ at $t = 2$.

3. If $V = \frac{4}{3}\pi r^3$, evaluate $\frac{dV}{dr}$ at $r = 5$.

4. A particle is moving in a straight line. Its distance, s metres, from a fixed point o after t seconds is given by $s = t^3 - 9t^2 + 15t + 2$.
Calculate:
(i) its velocity at any time t. **(ii)** its velocity after 6 seconds.
(iii) the distance of the particle from o when it is instantly at rest.
(iv) its acceleration after 4 seconds.

5. A car, starting at $t = 0$ seconds, travels a distance of s metres in t seconds where $s = 30t - \frac{9}{4}t^2$.
(i) Find the speed of the car after 2 seconds.
(ii) After how many seconds is the speed of the car equal to zero?
(iii) Find the distance travelled by the car up to the time its speed is zero.

6. The air resistance R to a body moving with speed v metres per second is given by $R = \dfrac{v^2}{100}$.
(i) Find the rate of change of the air resistance with respect to the speed.
(ii) Calculate this rate of change when $v = 16$ m/s.

7. A parachutist jumps out of an aeroplane. The distance, h metres, through which she falls after t seconds is given by $h = 10\,t - \dfrac{5t}{t+1}$. Find:
(i) the distance she falls in the first second.
(ii) her velocity after two seconds.

8. A particle moves in a straight line so that its distance s metres from a fixed point o at time t is given by $s = 1.5t^3 - 10.5t^2 - 4t + 10$.
(i) If its velocity after k seconds is 3.5 m/s, find the value of k.
(ii) If its acceleration after q seconds is 6 m/s^2, find the value of q.

9. The position, x metres, of a particle moving on the x-axis is given by $x = \cos 4t$ where t is in seconds. Find the velocity and the acceleration of the particle at $t = \dfrac{\pi}{4}$ seconds.

10. The distance, s metres, travelled by a car in t seconds after the brakes are applied is given by $s = 10t - t^2$. Show that its acceleration is constant. Find:
(i) the speed of the car when the brakes are applied.
(ii) the distance the car travels before it stops.

11. The distance, s metres, of an object from a fixed point in t seconds is given by $s = \dfrac{t+1}{t+3}$.
What is the speed of the object, in terms of t, at t seconds?
After how many seconds will the speed of the object be less than 0.02 m/s?

12. The equation $\theta = 3\pi + 20t - 2t^2$ gives the angle θ, in radians, through which a wheel turns in t seconds. Find:
(i) the rate of change of θ with respect to time t.
(ii) the time the wheel takes to come to rest.
(iii) the angle turned through in the last second of motion.

Rates of Change 2

Related rates of change

By convention, unless specified otherwise, the phrase 'rate of change' refers to the rate at which a variable is changing '**with respect to time**'. For example, if we are told the rate of change of h is 10, this means 'the rate of change of h with respect to time is 10'. In other words, we are given $\frac{dh}{dt} = 10$.

Related rates of change, using differentials, can be related using the chain rule.

For example:

$$\boxed{\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}}$$

$$\boxed{\frac{dV}{dr} = \frac{dV}{dt} \times \frac{dt}{dr}}$$

In many rate of change problems we will deal with 3 things:

> **1.** What we want to find. **2.** What we are given.
> **3.** What we need to complete the fraction (look for a link connecting the variables).

$$\text{Find} = (\text{Given}) \times \left(\begin{array}{c}\text{What we need to}\\ \text{complete the fraction}\end{array}\right)$$

Note: cm/s means centimetres per second, etc.

Example

The radius of a circle increases at 4 cm/s. What is the rate of increase of the area when the radius is 5 cm?

Solution:

The radius increases at 4 cm/s.

Thus, we are given $\frac{dr}{dt} = 4$ and asked to find $\frac{dA}{dt}$ when $r = 5$

Find = Given × Need

$$\frac{dA}{dt} = \frac{dr}{dt} \times \frac{dA}{dr}$$
$$= 4 \times 2\pi r$$
$$= 8\pi r$$
$$\left.\frac{dA}{dt}\right|_{r=5} = 8\pi(5) = 40\pi \text{ cm}^2/\text{s}$$

Link connecting A and r

$$A = \pi r^2$$
$$\frac{dA}{dr} = 2\pi r$$

Example

Air is pumped into a spherical balloon at the rate of 300 cm^3/s. When the radius of the balloon is 15 cm, calculate

(i) the rate at which its radius is increasing.
(ii) the rate at which its surface area is increasing.

Solution:

(i) Air is pumped in at a rate of 300 cm^3/s.

Thus, we are given $\frac{dV}{dt} = 300$ and asked to find $\frac{dr}{dt}$.

Find = Given × Need

$$\frac{dr}{dt} = \frac{dV}{dt} \times \frac{dr}{dV}$$

$$= 300 \times \frac{1}{4\pi r^2}$$

$$= \frac{300}{4\pi r^2} = \frac{75}{\pi r^2}$$

$$\left.\frac{dr}{dt}\right|_{r=15} = \frac{75}{\pi(15)^2} = \frac{75}{225\pi} = \frac{1}{3\pi} \text{ cm/s}$$

Link connecting V and r

$$V = \tfrac{4}{3}\pi r^3$$

$$\frac{dV}{dr} = 3 \times \tfrac{4}{3}\pi r^2$$

$$\frac{dV}{dr} = 4\pi r^2$$

$$\frac{dr}{dV} = \frac{1}{4\pi r^2}$$

(ii) Let the surface area = S.

We are given $\frac{dr}{dt} = \frac{1}{3\pi}$ and asked to find $\frac{dS}{dt}$ when $r = 15$.

Find = Given × Need

$$\frac{dS}{dt} = \frac{dr}{dt} \times \frac{dS}{dr}$$

$$= \frac{1}{3\pi} \times 8\pi r$$

$$= \frac{8r}{3}$$

$$\left.\frac{dS}{dt}\right|_{r=15} = \frac{8(15)}{3} = 40 \text{ cm}^2\text{/s}$$

Link connecting S and r

$$S = 4\pi r^2$$

$$\frac{dS}{dr} = 8\pi r$$

Exercise 13.6

Complete each of the following derivatives using the chain rule:

1. $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \text{——}$ 2. $\dfrac{dV}{dt} = \dfrac{dr}{dt} \times \text{——}$ 3. $\dfrac{dS}{dr} = \dfrac{dS}{dt} \times -$

4. $\dfrac{dA}{dt} = \dfrac{dr}{dt} \times \text{——}$ 5. $\dfrac{dh}{dt} = \dfrac{dV}{dt} \times \text{——}$ 6. $\dfrac{dA}{dr} = \dfrac{dA}{dt} \times -$

7. If $\dfrac{dx}{dt} = 5$ and $y = 2x^2 - 3x + 4$, find $\dfrac{dy}{dt}$ in terms of x.

8. If $y = (x^2 - 3x)^3$, find $\dfrac{dy}{dt}$ when $x = 2$, given $\dfrac{dx}{dt} = \dfrac{1}{2}$.

9. If $y = \left(\dfrac{x-1}{x}\right)^2$, find $\dfrac{dx}{dt}$ when $x = 2$, given $\dfrac{dy}{dt} = 4$.

10. The path of a projectile is given by $y = 2x - \dfrac{x^2}{20}$, $x \geqslant 0$.

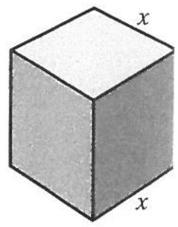
If $\dfrac{dx}{dt} = 4$, for all t, find $\dfrac{dy}{dt}$ when $x = 5$.

11. The radius of a circle is increasing at a rate of $\dfrac{1}{\pi}$ cm/s. Find the rate of increase of circumference.

12. The area of a square, of side x cm, is increasing at the rate of 8 cm^2/s. Find an expression, in t of x, for the rate of increase of the length of a side. Find this rate of increase when $x = 16$ cm

13. If $\dfrac{dV}{dt} = \dfrac{\pi}{2}$ and $V = \dfrac{4}{3}\pi r^3$, evaluate $\dfrac{dr}{dt}$ at $r = 2$.

14. A spherical snowball melts at the rate of 20 cm^3/h.
What is the rate of change of the radius when:
(i) the radius is r? **(ii)** the radius is 2 cm?
What is the rate of change of the surface area when the radius is 5 cm?



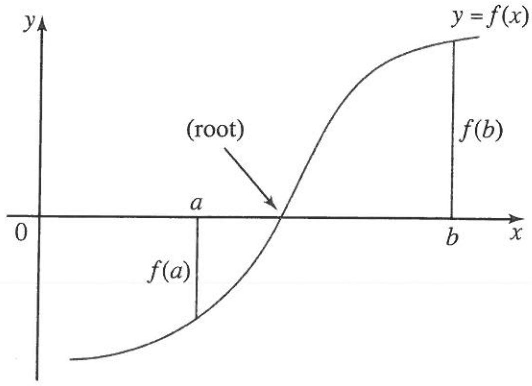
15. A metallic cube, of side length x cm, is being heated in a furnace. The side lengths are expanding at the rate of 0.2 cm/s. Find the rates at which the cube's surface area and the cube's volume are changing when $x = 5$ cm.

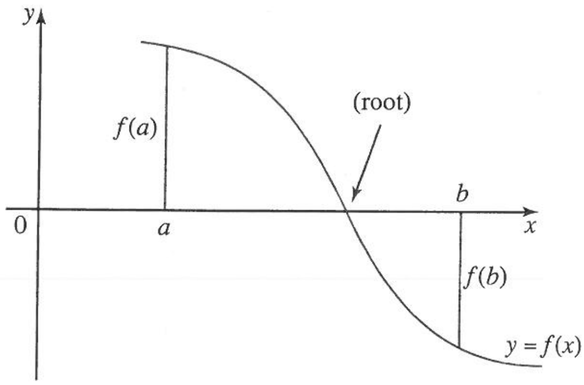
16. If a hemispherical bowl of radius 6 cm contains water to a depth of h cm, the volume of the water is $\frac{1}{3}\pi h^2(18 - h)$. Water is poured into the bowl at a rate of 4 cm^3/s. Find the rate at which the water level is rising when the depth is 2 cm.

17. A vessel is shaped such that when the depth of water is h cm, the volume is given by $v = \sqrt{5}$ If the height of the water is increasing at 18 cm/s, calculate the rate at which v is increasing v $h = 2$ cm.

Solving Cubic Equations Using the Newton–Raphson Method

Locating roots by a 'change of sign'

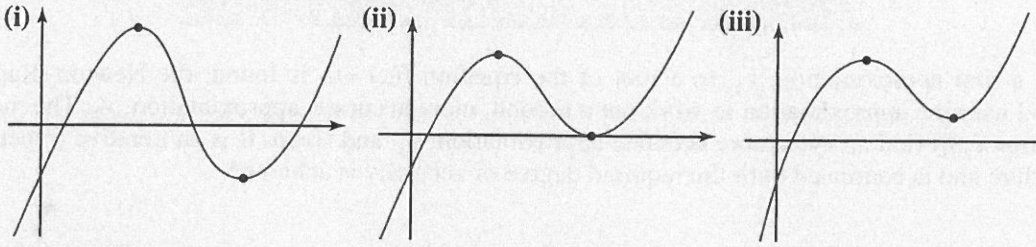




Suppose $f(x)$ is a continuous function between $x = a$ and $x = b$. If $f(a)$ and $f(b)$ have different signs, then somewhere between a and b there must be a root (solution) of the equation $f(x) = 0$.
By looking for a change of sign in the value of $f(x)$ between nearby values of x, the approximate location of the roots of the equation $f(x) = 0$ can be found.
If a cubic equation has real coefficients, then there must be either '**one real root**' or '**three real roots**'. The number of real roots of a cubic equation can be found by determining the turning points $\left(\frac{\mathrm{d}y}{\mathrm{d}x} = 0\right)$ of the curve of the cubic function.

Method for determining the number of real roots of the equation $ax^3 + bx^2 + cx + d = 0$:

1. Let $y = ax^3 + bx^2 + cx + d$.
2. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and then solve the equation $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ to find any turning points.
3. We consider three possible outcomes:



(i) 3 real different roots
Turning points are on **opposite** sides of the x-axis

(ii) 3 real roots, one repeated
One turning point **on** the x-axis

(iii) 1 real root
Turning points on the **same** side of the x-axis

Note: It is possible for a cubic equation to have a real triple root. This happens when the graph of the cubic function has a horizontal point of inflection (saddle point) on the x-axis. The graph has no turning points. The equation $(x - k)^3 = 0$ has a triple root, $x = k$. If the graph of $f(x)$ has no turning points, then $f(x) = 0$ has only one real root.

Example

(i) Show that the equation $x^3 + 2x - 5 = 0$ has only one real root.
(ii) Show that the equation $x^3 - 3x + 1 = 0$ has three real roots.

Solution:

(i) Let $f(x) = x^3 + 2x - 5$

$f'(x) = 3x^2 + 2$

Since $3x^2 + 2 > 0$ for all $x \in \mathbf{R}$, the graph of $f(x)$ is always increasing (no turning points).
Therefore, the graph of $f(x)$ cuts the x-axis at most once.
$\therefore$ $x^3 + 2x - 5 = 0$ has only one real root.

(ii) Let $f(x) = x^3 - 3x + 1$

$f'(x) = 3x^2 - 3$

$f''(x) = 6x$

$f'(x) = 0$ (max/min)

$\therefore$ $3x^2 - 3 = 0$

$x^2 - 1 = 0$

$x^2 = 1$

$x = \pm 1$

$f''(1) = 6(1) = 6 > 0$ $\therefore$ minimum turning point.
$f''(-1) = 6(-1) = -6 < 0$ $\therefore$ maximum turning point.

$x = 1$, $f(1) = (1)^3 - 3(1) + 1 = -1$
Thus a minimum turning point at $(1, -1)$.

$x = -1$, $f(-1) = (-1)^3 - 3(-1) + 1 = 3$
Thus a maximum turning point at $(-1, 3)$.

As both turning points are on **opposite** sides of the x-axis, the graph of $f(x)$ cuts the x-axis three times.
$\therefore$ $x^3 - 3x + 1 = 0$ has three real roots.

The Newton–Raphson Method

If x_n is an approximate solution of the equation $f(x) = 0$, then x_{n+1} is a better approximation, where:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

When a first approximation, x_1, to a root of the equation $f(x) = 0$, is found, the Newton–Raphso[n] method uses the approximation to work out a second, more accurate, approximation, x_2. The metho[d] then uses x_2 to find an even more accurate approximation, x_3, and so on. It is an iterative (repetitiv[e]) procedure and is continued until the required degree of accuracy is achieved.

Example

Show that the equation $1 - 3x - x^3 = 0$ has a root between 0 and 1.

Starting with $x_1 = 0$ as a first approximation of a real root of the equation $1 - 3x - x^3 = 0$, use two iterations of the Newton–Raphson method to find x_2 and x_3, the second and third approximations. Give your answers as fractions.

Solution:

Let $f(x) = 1 - 3x - x^3$

$f(0) = 1 - 3(0) - (0)^3 = 1 > 0$

$f(1) = 1 - 3(1) - (1)^3 = -3 < 0$

Since $f(x)$ changes sign between 0 and 1, the graph of the function $y = f(x)$ must cut the x-axis between 0 and 1.

$\therefore\ 1 - 3x - x^3 = 0$ has a root between 0 and 1.

$$f(x) = 1 - 3x - x^3 \quad \Rightarrow \quad f'(x) = -3 - 3x^2$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \text{(Newton–Raphson)}$$

$x_1 = 0$ (given)

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0 - \frac{f(0)}{f'(0)} = 0 - \frac{1 - 3(0) - (0)^3}{-3 - 3(0)^2} = 0 - \frac{1}{-3} = \frac{1}{3}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= \frac{1}{3} - \frac{f(\frac{1}{3})}{f'(\frac{1}{3})} = \frac{1}{3} - \frac{1 - 3(\frac{1}{3}) - (\frac{1}{3})^3}{-3 - 3(\frac{1}{3})^2} = \frac{1}{3} - \frac{1 - 1 - \frac{1}{27}}{-3 - \frac{1}{3}} = \frac{1}{3} - \frac{-\frac{1}{27}}{-\frac{10}{3}} = \frac{1}{3} - \frac{1}{90} = \frac{29}{30}$$

Example

Let $f(x) = x^3 - kx^2 + 9$, $\quad k \in \mathbf{R}$ and $k > 0$.

Taking $x_1 = 2$ as the first approximation of one of the roots of $f(x) = 0$, the Newton–Raphson method gives the second approximation as $\frac{13}{8}$.

Find the value of k.

Solution:

$$f(x) = x^3 - kx^2 + 9 \Rightarrow f'(x) = 3x^2 - 2kx$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \qquad \text{(Newton–Raphson method)}$$

$$\frac{13}{8} = 2 - \frac{f(2)}{f'(2)} \qquad \text{(put in } x_1 = 2 \text{ and } x_2 = \tfrac{13}{8}\text{)}$$

$$\frac{13}{8} = 2 - \frac{(2)^3 - k(2)^2 + 9}{3(2)^2 - 2k(2)}$$

$$\frac{13}{8} = 2 - \frac{8 - 4k + 9}{12 - 4k}$$

$$\frac{13}{8} = 2 - \frac{17 - 4k}{12 - 4k}$$

$$\frac{17 - 4k}{12 - 4k} = 2 - \frac{13}{8}$$

$$\frac{17 - 4k}{12 - 4k} = \frac{3}{8}$$

$$136 - 32k = 36 - 12k \qquad \text{(multiply both sides by } 8(12 - 4k)\text{)}$$

$$-20k = -100$$

$$20k = 100$$

$$k = 5$$

Exercise 13.7

1. Show that the equation $x^3 + x - 5 = 0$ has a root between 1 and 2. Taking $x_1 = 1$, as a first appro imation, use the Newton–Raphson method to find x_2, the second approximation.

2. Show that the equation $x^3 + 5x - 3 = 0$ has only one real root and that this root is between 0 and Taking $x_1 = 0.6$ as the first approximation of the real root of the equation, find, using t Newton–Raphson method, x_2, the second approximation, correct to two decimal places.

3. Show that the equation $x^3 - 12x + 6 = 0$ has three real roots.
   Show that one of these roots lies between 0 and 1. Taking $x_1 = \frac{1}{2}$ as a first approximation of root, apply the Newton–Raphson method once to obtain x_2, the second approximation, giv your answer as a fraction.

4. Given that $f(x) = x^3 - 3x^2 - 1$, show that the equation $f(x) = 0$ has only one real root and that this real root lies in the interval $3 < x < 4$.
Use two iterations of the Newton–Raphson method applied to $f(x) = 0$, with $x_1 = 3$, to find an approximation to the real root, giving your answer correct to three decimal places.

In each of the following, take x_1 as the first approximation of a real root of the given equation. Then, using one iteration of the Newton–Raphson method, find x_2, the second approximation. Write each answer as a fraction.

5. $x^3 - 5 = 0, \quad x_1 = 2$

6. $x^3 - 5x = 0, \quad x_1 = 2$

7. $x^3 - 3x^2 - 1 = 0, \quad x_1 = 3$

8. $x^3 - 5x^2 - x + 6 = 0, \quad x_1 = 1$

In each of the following, take x_1 as the first approximation of a real root of the given equation. Then, using two iterations of the Newton–Raphson method, find x_2 and x_3, the second and third approximations. Write your answers as fractions.

9. $x^3 - 4 = 0, \quad x_1 = 1$

10. $x^3 + 3x - 1 = 0, \quad x_1 = 0$

11. $x^3 - 7x + 5 = 0, \quad x_1 = 1$

12. $x^3 - 3x^2 + 3x - 3 = 0, \quad x_1 = 2$

13. Let $f(x) = a - x^3, \quad a \in \mathbf{R}$ and $a > 0$.
Taking $x_1 = 1$ as the first approximation to the real root of $f(x) = 0$, the Newton–Raphson method gives the second approximation as $x_2 = \frac{4}{3}$. Find the value of a.
Using this value of a, find x_3, the third approximation. Give your answer as a fraction.

14. Let $f(x) = x^3 - kx + 4, \quad k \in \mathbf{R}$ and $k > 0$.
Taking $x_1 = 2$ as the first approximation to a real root of $f(x) = 0$, the Newton–Raphson method gives the second approximation as $x_2 = 3$. Find the value of k.
Using this value of k, find x_3, the third approximation. Give your answer as a fraction.

15. The equation $x^3 + ax - 1 = 0$ is known to have a root close to $x = \frac{1}{2}$. When $x = \frac{1}{2}$ is used as the first approximation in the Newton–Raphson method, the second approximation is $\frac{5}{11}$. Find the value of a.

16. **(a)** Write $x + \dfrac{x^2 + 3x}{x + 1}$ as one fraction.

(b) Show that the Newton–Raphson method for approximating a root of the equation $x^3 + x - 6 = 0$, is given by $x_{n+1} = \dfrac{2x_n^3 + 6}{3x_n^2 + 1}$.

Taking 1.5 as a first approximation, apply the Newton–Raphson method once to obtain a better approximation, giving your answer correct to two decimal places.

17. Let $f(x) = x^3 - 3x^2 + k, \quad k \in \mathbf{R}$.
(i) Find the coordinates of the maximum, minimum and point of inflexion in terms of k.
(ii) Find the values of k for which the equation $f(x) = 0$ has three real roots.
(iii) If $k = 2$, use the Newton–Raphson method, with first approximation $x_1 = 3$, to find x_2, the second approximation. Write your answer in the form $\dfrac{p}{q}, \quad p, q \in \mathbf{N}$.