Example

If $f(x) = \dfrac{x^2}{x + \cos x}$, evaluate $f'\left(\dfrac{\pi}{2}\right)$.

Solution:

$$f(x) = \frac{x^2}{x + \cos x}$$

$$f'(x) = \frac{(x + \cos x)(2x) - x^2(1 - \sin x)}{(x + \cos x)^2} \qquad \text{(quotient rule)}$$

$$f'\left(\frac{\pi}{2}\right) = \frac{\left(\frac{\pi}{2} + \cos\frac{\pi}{2}\right)(\pi) - \left(\frac{\pi}{2}\right)^2\left(1 - \sin\frac{\pi}{2}\right)}{\left(\frac{\pi}{2} + \cos\frac{\pi}{2}\right)^2} \qquad \left(\begin{array}{l}\text{Don't simplify:} \\ \text{put in } x = \frac{\pi}{2}\end{array}\right)$$

$$= \frac{\left(\frac{\pi}{2} + 0\right)(\pi) - \left(\frac{\pi^2}{4}\right)(1 - 1)}{\left(\frac{\pi}{2} + 0\right)^2} \qquad \left(\cos\frac{\pi}{2} = 0,\ \sin\frac{\pi}{2} = 1\right)$$

$$= \frac{\left(\frac{\pi}{2}\right)(\pi) - \left(\frac{\pi^2}{4}\right)(0)}{\left(\frac{\pi}{2}\right)^2} = \frac{\frac{\pi^2}{2}}{\frac{\pi^2}{4}} = \frac{2\pi^2}{\pi^2} = 2$$

Exercise 12.3

Find $\dfrac{dy}{dx}$ if:

1. $y = \sin 4x$
2. $y = \cos 3x$
3. $y = \tan 2x$
4. $y = \sec 5x$
5. $y = -\operatorname{cosec} 6x$
6. $y = -2 \cot 4x$
7. $y = \sin(2x - 3)$
8. $y = \tan(3x + 2)$
9. $y = 2 \tan x + \sec x$
10. $y = x^2 \sin x$
11. $y = 3x \tan x$
12. $y = x^2 \cos 2x$
13. $y = \dfrac{\sin x}{x}$
14. $y = \dfrac{1}{1 - \sin x}$
15. $y = \dfrac{1 + \sin x}{\cos x}$
16. $y = \cos^3 x$
17. $y = \sin^2 4x$
18. $y = \tan^4 3x$
19. $y = (1 + \sin^2 x)^3$
20. $y = \sqrt{\sin x}$
21. $y = \sqrt{\cos 2x}$
22. $f(x) = \dfrac{\cos x + \sin x}{\cos x - \sin x}$. Show that $f'(x) = \dfrac{2}{1 - \sin 2x}$.

23. If $y = \cos 3x$, show that $\dfrac{d^2y}{dx^2} = -9y$.

24. If $y = 3\cos x + \sin x$, show that:

(i) $\cos x\left(\dfrac{dy}{dx}\right) + y\sin x - 1 = 0$ (ii) $\dfrac{d^2y}{dx^2} - 3\left(\dfrac{dy}{dx}\right) + 2y - 10\sin x = 0$.

25. If $f(x) = \sin x \cos x$, evaluate $f'\left(\dfrac{\pi}{4}\right)$.

26. If $y = \cos 2x + 2\sin x$, evaluate $\dfrac{dy}{dx}$ at $x = \dfrac{\pi}{6}$.

27. $f(x) = \dfrac{\sin x}{1 + \tan x}$. Evaluate $f'(0)$.

Implicit Differentiation

If $y = f(x)$, the variable y is given **explicitly** (clearly) in terms of x.
For example, $y = x^3 - 2x^2 + 5x - 4$ is an explicit function.
Some curves are defined by implicit functions, that is, functions which cannot be expressed in the forn $y = f(x)$.
For example, $x^2 + xy + y^3 = 7$ is an **implicit function**.
It cannot be written in the form $y = f(x)$.
It is for this reason that we must have a method for differentiating explicit functions.
An implicit function involving x and y can be differentiated with respect to x as it stands, using th chain rule.

Method for differentiating implicit functions:

1. Differentiate, term by term, on both sides with respect to x.
2. Bring all terms with $\dfrac{dy}{dx}$ to the left and bring all other terms to the right.
3. Make $\dfrac{dy}{dx}$ the subject of the equation.

It is useful to remember that, by the chain rule,

$$\frac{d}{dx}(y^2) = 2y\frac{dy}{dx} \quad \text{and} \quad \frac{d}{dx}(y^3) = 3y^2\frac{dy}{dx}$$

as y is considered as a function of x.

$$\frac{d}{dx}(y^n) = ny^{n-1}\left(\frac{dy}{dx}\right)$$

Example

Given that $2x^3 + 3xy^2 - y^3 + 6 = 0$, evaluate $\frac{dy}{dx}$ at the point $(-1, 1)$.

Solution:

(use product rule here)

$$2x^3 + 3xy^2 - y^3 + 6 = 0$$

$$6x^2 + 3\left[x \,.\, 2y\frac{dy}{dx} + y^2(1)\right] - 3y^2\frac{dy}{dx} = 0$$

$$6x^2 + 6xy\frac{dy}{dx} + 3y^2 - 3y^2\frac{dy}{dx} = 0$$

$$6xy\frac{dy}{dx} - 3y^2\frac{dy}{dx} = -6x^2 - 3y^2$$

$$\frac{dy}{dx}(6xy - 3y^2) = -6x^2 - 3y^2$$

(divide each term by -3)

$$\frac{dy}{dx} = \frac{-6x^2 - 3y^2}{6xy - 3y^2} = \frac{2x^2 + y^2}{y^2 - 2xy} = \frac{2x^2 + y^2}{y(y - 2x)}$$

$$\left.\frac{dy}{dx}\right|_{\substack{x=-1\\y=1}} = \frac{2(-1)^2 + (1)^2}{1(1+2)} = \frac{3}{3} = 1$$

Note: To evaluate $\frac{dy}{dx}$ we used both coordinates of the point.

Exercise 12.4

For each of the following curves, express $\frac{dy}{dx}$ in terms of x and y:

1. $x^2 + y^2 = 4$
2. $x^2 + 2y - y^2 = 5$
3. $x^2 - 6y^3 + y = 0$
4. $x^2 + y^2 - 4x - 6y + 9 = 0$
5. $x^2 + xy + y^2 = 13$
6. $x^2 + 3xy + 2y^2 = 6$
7. $x^2y - 5x = 2$
8. $xy^2 + x^2 = 2$
9. $x^2y + xy^2 = 2$

Find the value of $\frac{dy}{dx}$ at the point specified:

10. $x^2 + y^2 = 25$ at the point $(3, -4)$
11. $x^2 + xy + 2y^2 = 28$ at the point $(2, -4)$
12. $x^2 + 4xy - 2y^2 - 8 = 0$ at the point $(0, 2)$
13. $x^3 + y^2 + 3x^2y = 21$ at the point $(2, 1)$
14. Find the slope of the tangent to the curve $y^2 + 3xy + 2x^2 = 6$ at the point $(1, 1)$.
15. Find the slope of the tangent to the curve $x \sin y + y^2 = 1 + \frac{\pi^2}{4}$ at the point $\left(1, \frac{\pi}{2}\right)$.

Note: $\frac{d}{dx}(\sin y) = \cos y\frac{dy}{dx}$.

Parametric Differentiation

If x and y are each expressed in terms of a third variable, t say (or θ), called the **parameter**, then $x = f(t)$ and $y = g(t)$ give the parametric forms of the equation relating to x and y respectively.

To find $\frac{dy}{dx}$ do the following:

1. Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$, separately. **2.** Use $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

Example

(i) If $x = 4t^3$ and $y = (1 + 3t^2)^2$, express $\frac{dy}{dx}$ in terms of t.

Hence, or otherwise, evaluate $\frac{dy}{dx}$ when $t = -1$.

(ii) Let $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$; show that $\frac{dy}{dx} = \tan\theta$.

$$\left[a \neq 0, -\pi < \theta < \pi \quad \text{and} \quad \theta \neq \pm\frac{\pi}{2}\right]$$

Solution:

$x = 4t^3$

$$\frac{dx}{dt} = 12t^2$$

$y = (1 + 3t^2)^2$

(chain rule)

$$\frac{dy}{dt} = 2(1 + 3t^2)^1(6t) = 12t(1 + 3t^2)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{12t(1 + 3t^2)}{12t^2} = \frac{1 + 3t^2}{t}$$

$$\left.\frac{dy}{dx}\right|_{t=-1} = \frac{1 + 3(-1)^2}{-1} = \frac{1 + 3}{-1} = \frac{4}{-1} = -4$$

(ii) $x = a(\cos\theta + \theta\sin\theta)$

$$\frac{dx}{d\theta} = a(-\sin\theta + \theta.\cos\theta + \sin\theta.1)$$
$$= a(-\sin\theta + \theta\cos\theta + \sin\theta)$$
$$= a\theta\cos\theta$$

$y = a(\sin\theta - \theta\cos\theta)$

$$\frac{dy}{d\theta} = a[\cos\theta - (\theta.-\sin\theta + \cos\theta.1)]$$
$$= a(\cos\theta + \theta\sin\theta - \cos\theta)$$
$$= a\theta\sin\theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

Exercise 12.5

Find $\frac{dy}{dx}$, in terms of t, if:

1. $x = 2t,\quad y = t^2$
2. $x = 2t + 3,\quad y = 2t^3$
3. $x = t^2 + 1,\quad y = t^3$
4. $x = 3t^4,\quad y = 2t^2 + 5$
5. $x = t(1 - t),\quad y = t(1 - t^2)$
6. $x = 6t + 5,\quad y = (2t - 1)^3$
7. $x = \frac{1}{t},\quad y = t^2 + 4t$
8. $x = 2\sqrt{t},\quad y = 5t + 4$
9. $x = 1 + \frac{1}{t},\quad y = t + \frac{1}{t}$
10. $x = \frac{t^2}{1 + t^3},\quad y = \frac{t^3}{1 + t^3}$
11. $x = \frac{t - 2}{t + 1}$ and $y = \frac{t + 2}{t + 1}$. If $\frac{dy}{dx} = k$, find the value of k.
12. If $x = \frac{2}{t}$ and $y = 3t^2 - 1$, express $\frac{dy}{dx}$ in terms of t. Evaluate $\frac{dy}{dx}$ at the point (2, 2).
13. If $x = \frac{3t - 1}{t}$ and $y = \frac{t^2 + 4}{t}$, express $\frac{dy}{dx}$ in terms of t.

    Find the values of t for which $\frac{dy}{dx} = 0$.
14. If $x = 2t + \sin 2t$ and $y = \cos 2t$, show that $\frac{dy}{dx} = -\tan t$.
15. If $x = \sec\theta$ and $y = \tan\theta$, show that $\frac{dy}{dx} = \operatorname{cosec}\theta$.
16. If $x = k(\theta - \sin\theta)$ and $y = k(1 - \cos\theta)$, $k \in \mathbf{R}$, find $\frac{dy}{dx}$.
17. Given $y = \sin\theta\cos\theta - \theta$ and $x = 2\cos\theta$, show that **(i)** $\frac{dy}{d\theta} = -2\sin^2\theta$ **(ii)** $\frac{dy}{dx} = \sin\theta$.
18. If $x = 3\cos\theta - 4\sin\theta$ and $y = 4\cos\theta + 3\sin\theta$, evaluate $\frac{dy}{dx}$ at $\theta = \frac{\pi}{2}$.

19. If $x = \sin\theta$ and $y = \sin n\theta$, where $n \in \mathbf{R}$, show that $(1 - x^2)\left(\dfrac{dy}{dx}\right)^2 - n^2(1 - y^2) = 0$.

20. $x = k(1 + \cos\theta)$, $\quad y = 2k\sin^2\theta$, where $0 \leqslant \theta \leqslant \pi$ and k is a positive constant.

(i) Find $\dfrac{dy}{dx}$ in the form $p\cos\theta$ where $p \in \mathbf{Z}$.

(ii) Find, in terms of k, the coordinates of the point q where $\theta = \tan^{-1}\dfrac{\sqrt{7}}{3}$.

Differentiation of Inverse Trigonometric Functions

The rules for differentiating also apply to inverse trigonometric functions.
The following are in the tables on page 41, but they are shown only for x.
The chain rule is used throughout, assuming u is a function of x.

Replace a with 1, x with u, and always multiply by $\dfrac{du}{dx}$.

Basic rule (page 41 tables)	
$f(x)$	$f'(x)$
$\sin^{-1}\dfrac{x}{a}$	$\dfrac{1}{\sqrt{a^2 - x^2}}$
$\tan^{-1}\dfrac{x}{a}$	$\dfrac{a}{a^2 + x^2}$

Chain rule	
$f(u)$	$f'(u)\cdot\dfrac{du}{dx}$
$\sin^{-1} u$	$\dfrac{1}{\sqrt{1 - u^2}}\cdot\dfrac{du}{dx}$
$\tan^{-1} u$	$\dfrac{1}{1 + u^2}\cdot\dfrac{du}{dx}$

Note: The derivative of $\cos^{-1} u$ is **not** in the syllabus.

Example

(i) If $y = \tan^{-1}\left(\dfrac{x}{1 + x}\right)$, show that $\dfrac{dy}{dx} = \dfrac{1}{2x^2 + 2x + 1}$, $\quad x \neq -1$.

(ii) Given $y = \sin^{-1}(3x - 1)$, calculate the value of $\dfrac{dy}{dx}$ at $x = \frac{1}{3}$.

Solution:

$$y = \tan^{-1}\left(\frac{x}{1 + x}\right)$$

(quotient rule)

$$\frac{dy}{dx} = \frac{1}{1 + \left(\dfrac{x}{1 + x}\right)^2}\cdot\left(\frac{(1 + x)(-1) - (x)(1)}{(1 + x)^2}\right)$$

$$\boxed{\begin{gathered} y = \tan^{-1} u \\ \frac{dy}{dx} = \frac{1}{1 + u^2}\cdot\frac{du}{dx} \end{gathered}}$$

$$= \frac{1}{1 + \frac{x^2}{(1+x)^2}} \cdot \left(\frac{1+x-x}{(1+x)^2}\right)$$

$$= \frac{(1+x)^2}{(1+x)^2 + x^2} \cdot \frac{1}{(1+x)^2}$$

$\left(\text{multiply the top and bottom of the first fraction by } (1+x)^2\right)$

$$= \frac{1}{(1+x)^2 + x^2}$$

$$= \frac{1}{1 + 2x + x^2 + x^2}$$

$$= \frac{1}{2x^2 + 2x + 1}$$

(ii)

$$y = \sin^{-1}(3x - 1)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(3x-1)^2}} \cdot (3)$$

$$= \frac{3}{\sqrt{1-(3x-1)^2}}$$

$$\boxed{\begin{gathered} y = \sin^{-1} u \\ \frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx} \end{gathered}}$$

$$\left.\frac{dy}{dx}\right|_{x=\frac{1}{3}} = \frac{3}{\sqrt{1-(0)^2}} = \frac{3}{\sqrt{1}} + \frac{3}{1} = 3$$

Exercise 12.6

Find $\frac{dy}{dx}$ for each of the following:

1. $y = \sin^{-1} 2x$ **2.** $y = \tan^{-1} 3x$ **3.** $y = \sin^{-1}(x - 1)$

4. $y = \tan^{-1}(2x + 1)$ **5.** $y = \tan^{-1} x^2$ **6.** $y = \sin^{-1} 2x^3$

7. $y = (\sin^{-1} 5x)^2$ **8.** $y = \tan^{-1}\left(\frac{x}{3}\right)$ **9.** $y = \sin^{-1}\left(\frac{x}{2}\right)$

10. $y = \sin^{-1}(\cos x)$ **11.** $y = x \sin^{-1} x$ **12.** $y = 6x \tan^{-1} 2x$

13. Given $y = \sin^{-1}(4x - 1)$, calculate the value of $\frac{dy}{dx}$ at $x = \frac{1}{4}$.

14. Given $y = \tan^{-1}\left(\frac{1}{x}\right)$, show that $\frac{dy}{dx} = -\frac{1}{1+x^2}$.

15. Given $y = \tan^{-1}(\cos x)$, calculate the value of $\frac{dy}{dx}$ at $x = \frac{\pi}{6}$.

16. If $y = \tan^{-1}\left(\frac{x}{a}\right)$, show that $\frac{dy}{dx} = \frac{a}{a^2 + x^2}$.

17. Explain why $p\sqrt{1-q} = \sqrt{p^2 - p^2 q}$, $p, q \in \mathbf{R}$.

If $y = \sin^{-1}\left(\frac{x}{a}\right)$, show that $\frac{dy}{dx} = \frac{1}{\sqrt{a^2 - x^2}}$.

18. $f(x) = \frac{1}{x}\sin^{-1}\left(\frac{1}{x}\right)$. Show that $f'(\sqrt{2}) = -\frac{1}{2} - \frac{\pi}{8}$.

19. If $y = \tan^{-1} x$, show that $\frac{d^2y}{dx^2}(1 + x^2) + 2x\frac{dy}{dx} = 0$.

20. If $u = \frac{1+x}{1-x}$, show that $\frac{du}{dx} = \frac{2}{(1-x)^2}$.

Hence, if $y = \tan^{-1}\left(\frac{1+x}{1-x}\right)$, find $\frac{dy}{dx}$.

Verify that $2x\left(\frac{dy}{dx}\right)^2 + \frac{d^2y}{dx^2} = 0$.

21. Explain why $\sqrt{a} = \frac{a}{\sqrt{a}}$, $a \in \mathbf{R}$, $a \neq 0$.

Given $y = \sin^{-1} x + x\sqrt{1-x^2}$, show that $\frac{dy}{dx} = 2\sqrt{1-x^2}$.

Differentiation of Exponential Functions

The rules for differentiating apply also to exponential functions.
Exponent is another word for index. A function such as $y = 2^x$, in which the variable occurs as an index, is called 'an exponential function'.
The function $y = e^x$ is called '**the exponential function**' or '**natural exponential function**'.
e is an irrational constant whose value is 2.71828 correct to six significant figures.
e^x is the only basic function which is its own derivative. That is:

$$\text{If } y = e^x, \quad \frac{dy}{dx} = e^x.$$

Note: The positive number e behaves just like other positive numbers such as 2 or 5.
e^x obeys all the usual laws of indices or exponents.

Using the chain rule:

Suppose u is a function of x.

$$\text{If } \quad y = e^u$$

$$\text{then } \quad \frac{dy}{dx} = e^u \cdot \frac{du}{dx}.$$

occurs as an

on'.

Example

Find $\frac{dy}{dx}$ if **(i)** $y = e^{x^2-3x}$ **(ii)** $y = \frac{2}{e^{3x}}$ **(iii)** $y = e^{\sin 2x}$ **(iv)** $y = \frac{x}{e^{2x}}$

Solution:

(i)
$$y = e^{x^2-3x}$$
$$\frac{dy}{dx} = e^{x^2-3x}(2x-3)$$
$$= (2x-3)e^{x^2-3x}$$

(ii)
$$y = \frac{2}{e^{3x}} = 2e^{-3x}$$
$$\frac{dy}{dx} = 2e^{-3x}(-3)$$
$$= -6e^{-3x} = -\frac{6}{e^{3x}}$$

(iii)
$$y = e^{\sin 2x}$$
$$\frac{dy}{dx} = e^{\sin 2x}(\cos 2x)(2)$$
$$= (2\cos 2x)e^{\sin 2x}$$

(iv)
$$y = \frac{x}{e^{2x}} = xe^{-2x}$$
(use the product rule)
$$\frac{dy}{dx} = xe^{-2x}(-2) + e^{-2x}(1)$$
$$= -2xe^{-2x} + e^{-2x}$$
$$= e^{-2x}(1-2x) = \frac{1-2x}{e^{2x}}$$

Note: The quotient rule could also be used.

Example

If $y = xe^{-x}$, show that $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$.

Solution:

$$y = xe^{-x}$$
$$\frac{dy}{dx} = x[e^{-x}(-1)] + e^{-x}(1) \qquad \text{(product rule)}$$
$$= -xe^{-x} + e^{-x}$$
$$= e^{-x}(1-x)$$

$$\frac{d^2y}{dx^2} = e^{-x}(-1) + (1-x)[e^{-x}(-1)] \qquad \text{(product rule, again)}$$
$$= -e^{-x} + (1-x)(-e^{-x})$$
$$= -e^{-x} - e^{-x} + xe^{-x}$$
$$= xe^{-x} - 2e^{-x}$$
$$= e^{-x}(x-2)$$

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y$$
$$= e^{-x}(x-2) + 2[e^{-x}(1-x)] + (xe^{-x})$$
$$= xe^{-x} - 2e^{-x} + 2e^{-x} - 2xe^{-x} + xe^{-x}$$
$$= e^{-x}(x - 2 + 2 - 2x + x) = e^{-x}(0) = 0.$$

Exercise 12.7

Find $\frac{dy}{dx}$ for each of the following:

1. $y = e^{4x}$ **2.** $y = 2e^{3x}$ **3.** $y = e^{x^2}$ **4.** $y = e^{x^2-5x}$

5. $y = e^{4x^2}$ **6.** $y = e^{-x}$ **7.** $y = \frac{5}{e^{2x}}$ **8.** $y = \frac{2}{e^{x^2}}$

9. $y = e^{\sin x}$ **10.** $y = e^{\cos 2x}$ **11.** $y = e^{4\tan x}$ **12.** $y = e^{x\sin x}$

13. $y = xe^x$ **14.** $y = x^2e^{5x}$ **15.** $y = e^{2x}\cos x$ **16.** $y = e^{-x^2}\sin x$

17. $y = \frac{x^2}{e^{2x}}$ **18.** $y = (3 + e^{x^2})^4$ **19.** $y = \frac{1}{3 - e^{2x^2}}$ **20.** $y = \sqrt{1 - 2e^{4x}}$

21. If $f(x) = \frac{1 + e^x}{1 - e^x}$, show that $f'(x) = \frac{2e^x}{(1 - e^x)^2}$.

22. If $f(\theta) = e^{1 + \sin\theta}$, evaluate **(i)** $f'(0)$ **(ii)** $f''($

23. If $y = e^{2x}$, show that $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$.

24. If $y = xe^{-2x}$, show that $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$.

25. If $y = e^x \sin x$, show that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$.

26. Given that $y = x + \sin^{-1} x$, show that $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + x = 0$.

27. If $y = e^{kx}$, find the values of $k \in \mathbf{R}$ for which $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$.

28. If $y = e^{2t}$ and $x = e^t$, show that $\frac{dy}{dx} = 2e^t$.

29. If $y = te^t$ and $x = t^2e^t$, show that $\frac{dy}{dx} = \frac{t + 1}{t(t + 2)}$.

30. Given $y = e^\theta \cos\theta$ and $x = e^\theta \sin\theta$, where $-\frac{3\pi}{4} < \theta < \frac{\pi}{4}$, show that $\left(\frac{dy}{d\theta}\right)^2 + \left(\frac{dx}{d\theta}\right)^2 = 2e^{2\theta}$.

Evaluate $\frac{dy}{dx}$ at $\theta = \frac{\pi}{2}$.

31. If $y = e^{-nx}\cos kx$, $n, k \in \mathbf{R}$, show that $\frac{d^2y}{dx^2} + 2n\frac{dy}{dx} + (n^2 + k^2)y = 0$.

Differentiation of Natural Logarithmic Functions

Logarithms to the base e are called '**natural logarithms**'.
The notation $\ln x$ is used as an abbreviation of $\log_e x$.
The function $y = \ln x$ is the inverse function of $y = e^x$
(exponents and logs are inverse functions of each other).

Note: $\log_e x$ or $\ln x$ is defined only for $x > 0$.

Natural logarithms obey the same laws as logarithms to any other base.

Laws of Logs:	$\ln ab = \ln a + \ln b$	$\ln \dfrac{a}{b} = \ln a - \ln b$	$\ln a^n = n \ln a$

Using the laws of logs before differentiating can simplify the work.

The following is worth remembering when evaluating the derivatives of natural logarithmic functions:

$$\ln e^k = k, \qquad \text{for any } k \in \mathbf{R}.$$

For example,

$\ln 1 = \ln e^0 = 0,$ $\ln e = \ln e^1 = 1,$ $\ln e^2 = 2,$ $\ln\sqrt{e} = \ln e^{1/2} = \frac{1}{2}.$

The rules for differentiating also apply to natural logarithmic functions.

Suppose u is a function of x.

If $y = \ln u$

then $\dfrac{dy}{dx} = \dfrac{1}{u} \cdot \dfrac{du}{dx}.$

Example

Find $\dfrac{dy}{dx}$ if **(i)** $y = \ln(x^2+1)$ **(ii)** $y = \ln(\sin x)$ **(iii)** $y = \ln\sqrt{x^2-3}$ **(iv)** $y = x \ln x$.

Solution:

(i) $y = \ln(x^2+1)$

$$\frac{dy}{dx} = \frac{1}{x^2+1} \cdot 2x = \frac{2x}{x^2+1}$$

(ii) $y = \ln(\sin x)$

$$\frac{dy}{dx} = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x} = \cot x$$

(iii) $y = \ln\sqrt{x^2-3}$

$y = \ln(x^2-3)^{1/2} = \frac{1}{2}\ln(x^2-3)$

(using $\ln a^n = n \ln a$)

$$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{x^2-3} \cdot 2x = \frac{x}{x^2-3}$$

(iv) $y = x \ln x$

$$\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \ln(x)(1)$$

(using the product rule)

$= 1 + \ln x$

Exercise 12.8

Find $\frac{dy}{dx}$ for each of the following:

1. $y = \ln 5x$
2. $y = \ln(2x + 3)$
3. $y = \ln(x^2 + 3$
4. $y = \ln(\cos x)$
5. $y = \ln\left(\frac{1}{x}\right)$
6. $y = \ln(e^x + 2$
7. $y = \ln(\sin 2x)$
8. $y = \ln(\tan 3x)$
9. $y = \ln(e^{2x})$
10. $y = x \ln x^2$
11. $y = x^3 \ln(x + 1)$
12. $y = x^2 \ln 4x$

Use the rules of logarithms, or otherwise, to find $\frac{dy}{dx}$ for each of the following:

13. $y = \ln\left(\frac{2x}{x+1}\right)$
14. $y = \ln(2x + 3)^2$
15. $y = \ln\left(\frac{1}{e^x}\right)$
16. $y = \ln\sqrt{1 + x^2}$
17. $y = \ln\sqrt{\sin x}$
18. $y = \ln\sqrt{\frac{x}{1+}}$
19. If $f(x) = \ln(e^x \cos x)$, show that $f'(x) = 1 - \tan x$.
20. If $y = \ln(\sec x + \tan x)$, show that $\frac{dy}{dx} = \sec x$.
21. If $f(x) = x^2 \ln x$, evaluate **(i)** $f'(e)$ **(ii)** $f'(1)$.
22. Given $f(x) = \ln\left(\frac{1 + \cos x}{1 - \cos x}\right)$, show that $f'(x) = -2 \operatorname{cosec} x$.
23. If $f(x) = \ln(\ln x)$, evaluate $f'(e)$.
24. If $f(x) = \ln\left(\frac{e^x}{1 + e^x}\right)$ evaluate $f'(0)$.
25. If $y = \frac{\ln x}{x}$, show that $\frac{dy}{dx} = \frac{1 - \ln x}{x^2}$.

    Evaluate $\frac{d^2y}{dx^2}$ at $x = e$.
26. Given $f(x) = e^x \ln x$, $x > 0$, evaluate $f''(1)$.
27. Given $y = \ln(t + 1)$ and $x = 1 + \ln t$, express $\frac{dy}{dx}$ in terms of t.
28. If $y = e^{t+1}$ and $x = e^t$, find the value of $\ln\left(\frac{dy}{dx}\right)$.
29. If $y = \ln t$ and $x + \frac{1}{2}\left(t + \frac{1}{t}\right)$, show that $\frac{dy}{dx} = \frac{2t}{t^2 - 1}$.
30. If $x = \ln\left(\frac{e^t}{1 + e^t}\right)$ and $y = \ln\left(\frac{1 + e^t}{e^t}\right)$, $t \in \mathbf{R}$, evaluate $\frac{dy}{dx}$.
31. Given $y = x \ln(x^2)$, show that $x\left(\frac{dy}{dx}\right) - 2x = y$.

32. Using $\ln \frac{a}{b} = \ln a - \ln b$, or otherwise, show that if $y = \ln\left(\frac{1+x}{1-x}\right)$,

(i) $(1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x} = 2$ **(ii)** $\left(\frac{2x}{1-x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0.$

33. Factorise $a^x + a^{2x}$. If $y = \ln(1+e^x)$, show that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \frac{\mathrm{d}y}{\mathrm{d}x}$.

34. If $y = \ln e^{-x}\sqrt{\frac{1+2x}{1-2x}}$, show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1+4x^2}{1-4x^2}$.

Find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $x = -1$.

(Hint: $\ln \frac{ab}{c} = \ln a + \ln b - \ln c$)

Logarithmic Differentiation

Functions of the form 2^x, x^x or $3^{\sin x}$ are differentiated using **'logarithmic differentiation'**.

The method involves three steps:

1. Take natural logs of both sides and use the fact that $\ln a^x = x \ln a$.
2. Differentiate both sides with respect to x, using implicit differentiation.
3. Multiply both sides by y to get $\frac{\mathrm{d}y}{\mathrm{d}x}$ on its own.

Example

Differentiate **(i)** 2^x **(ii)** x^x with respect to x.

Solution:

(i) Let

$$\begin{aligned} y &= 2^x \\ \ln y &= \ln 2^x \\ \ln y &= x \ln 2 \\ \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \ln 2 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= y \ln 2 \\ &= 2^x \ln 2 \end{aligned}$$

(ii) Let

$$\begin{aligned} y &= x^x \\ \ln y &= \ln x^x \\ \ln y &= x \ln x \\ \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= x\left(\frac{1}{x}\right) + \ln x(1) \end{aligned}$$

(using the product rule)

$$\begin{aligned} \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= 1 + \ln x \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= y(1 + \ln x) \\ &= x^x(1 + \ln x) \end{aligned}$$

Exercise 12.9

Use logarithmic differentiation to find the derivative of each of the following:

1. 3^x

2. 5^x

3. 3^{2x}

4. 4^{3x+1}

5. $2^{\sin x}$

6. $2^{\ln x}$

7. $(\sin x)^x$

8. $2^x x^2$

9. If $f(x) = x4^x$, evaluate $f'(1)$.

10. If $y = a^x$, $a > 0$, $a \in \mathbf{R}$, show that $\dfrac{dy}{dx} = a^x \ln a$.

11. If $x^y = e^x$, show that $\dfrac{dy}{dx} = \dfrac{x - y}{x \ln x}$ or $\dfrac{\ln x - 1}{(\ln x)^2}$.

4. 4^{3x+1}

8. $2^x x^2$

CHAPTER 13

DIFFERENTIAL CALCULUS 3
APPLICATIONS OF DIFFERENTIATION

Finding the Equation of a Tangent to a Curve at a Point on the Curve

$$\frac{dy}{dx} = \text{the slope of a tangent to a curve at any point on the curve}$$

To find the equation of a tangent to a curve at a given point, (x_1, y_1), on the curve, do the following:

Step 1: Find $\frac{dy}{dx}$.

Step 2: Evaluate $\left.\frac{dy}{dx}\right|_{x=x_1}$ [this gives m, the slope of the tangent]

(If the equation of the curve is given implicitly, use $\left.\frac{dy}{dx}\right|_{\substack{x=x_1\\y=y_1}}$)

Step 3: Use m (from step 2) and the given point (x_1, y_1) in the equation: $(y - y_1) = m(x - x_1)$.

Note: Sometimes only the value of x is given. When this happens, substitute the value of x into the original function to find y for step 3.

Example

(i) Find the equation of the tangent to the curve $x^2 + xy + y^2 = 3$ at the point $(1, 1)$.
(ii) Find the equation of the tangent to the curve defined by:
$x = t - 2\cos t$ and $y = 2\sin t - 2\cos t$ at the point where $t = 0$.

Solution:

(i)

$$x^2 + xy + y^2 = 3 \qquad \text{(implicit differentiation required)}$$

$$2x + x\frac{dy}{dx} + y(1) + 2y\frac{dy}{dx} = 0 \qquad \text{(use the product rule on } xy\text{)}$$

$$x\frac{dy}{dx} + 2y\frac{dy}{dx} = -2x - y$$

$$\frac{dy}{dx}(x + 2y) = -2x - y$$

$$\frac{dy}{dx} = \frac{-2x - y}{x + 2y}$$

$$\left.\frac{dy}{dx}\right|_{\substack{x=1\\y=1}} = \frac{-2(1) - (1)}{1 + 2(1)} = \frac{-3}{3} = -1$$

At the point $(1, 1)$ the slope $= -1$.
Equation of the tangent at the point $(1, 1)$:

$$(y - 1) = -1(x - 1)$$
$$y - 1 = -x + 1$$
$$x + y - 2 = 0$$

(ii) $x = t - 2\cos t$ $\quad$ $y = 2\sin t - 2\cos t$

(parametric differentiation required)

$$x = t - 2\cos t$$
$$\frac{dx}{dt} = 1 - 2(-\sin t)$$
$$= 1 + 2\sin t$$

$$y = 2\sin t - \cos t$$
$$\frac{dy}{dt} = 2\cos t - 2(-\sin t)$$
$$= 2\cos t + 2\sin t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2\cos t + 2\sin t}{1 + 2\sin t}$$

$$\left.\frac{dy}{dx}\right|_{t=0} = \frac{2\cos(0) + \sin(0)}{1 + 2\sin(0)}$$
$$= \frac{2(1) + 2(0)}{1 + 2(0)}$$
$$= \frac{2}{1}$$
$$= 2$$

$t = 0$

$x = t - 2\cos t$	$y = 2\sin t - 2\cos t$
$= 0 - 2(1)$	$= 2(0) - 2(1)$
$= -2$	$= -2$

Thus, the point $(-2, -2)$ is on the curve at $t = 0$.
Equation of the tangent at $t = 0$:

$$(y + 2) = 2(x + 2)$$
$$y + 2 = 2x + 4$$
$$2x - y + 2 = 0$$

Sometimes we are given the value of $\frac{dy}{dx}$ and asked to find unknown coefficients.

Example

The slope of the tangent to the curve $y = ax^3 + bx + 4$ is 21 at the point $(2, 14)$ on the curve. Find the value of a and the value of b.

Solution:

$$y = ax^3 + bx + 4$$
$$\frac{dy}{dx} = 3ax^2 + b = 21 \quad \text{(when } x = 2\text{)}$$
$$3a(2)^2 + b = 21 \quad \text{(put in } x = 2\text{)}$$
$$12a + b = 21 \quad ①$$

Given: $(2, 14)$ is on the curve

Thus, $14 = a(2)^3 + b(2) + 4$

$$14 = 8a + 2b + 4$$
$$8a + 2b = 10 \quad ②$$

Solving the simultaneous equations ① and ② gives $a = 2$ and $b = -3$.

Exercise 13.1

Find the equation of the tangent to the curve at the indicated point:

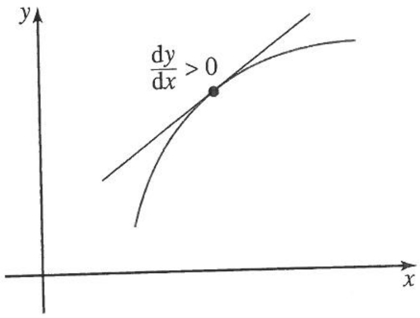
1. $y = 3 + 2x - x^2$ at $(2, 3)$
2. $y = x^3 - 2x^2 - 4x + 1$ at $(-1, 2)$
3. $y = (2x + 3)^3$ at $(-1, 1)$
4. $y = \dfrac{6x - 3}{4x + 2}$ at $(1, \frac{1}{2})$
5. $x^2 + y^2 - 10y = 0$ at $(4, 2)$
6. $y^3 - xy - 6x^3 = 0$ at $(1, 2)$
7. $x = 3t^2, \quad y = 6t$ at $t = 1$
8. $y = \ln x$ at $x = 1$
9. $y = 2 \cos x + \sin x$ at $(0, 2)$
10. $y = \tan^{-1} x$ at $x = 0$
11. Find the equation of the tangent to the curve $y = x + e^{2x}$ at the point where $x = 0$.
12. Find the equation of the tangent to the curve $x = e^t + t, \quad y = e^{3t} - 2t$, at the point where $t = 0$.
13. Find the equation of the tangent to the curve $x = 2 + \ln t, \quad y = t^3$, at the point $(2, 1)$.
14. Find the equation of the tangent to the curve $x = (1 + t)^2, \quad y = (1 - t)^2$, at the point where $y = x$.
15. Find the equations of the two tangents to the curve $y^2 + 3xy + 4x^2 = 14$ at the points where $x = 1$.
16. Find the equation of the tangent to the curve $x = 4 \cos \theta + 3 \sin \theta + 2, \quad y = 3 \cos \theta - 4 \sin \theta - 1$, at the point where $\theta = \dfrac{\pi}{2}$.
17. Find the coordinates of the points on the curve $y = \dfrac{x}{1 + x}$ at which the tangents to the curve are parallel to the line $x - y + 8 = 0$. Find the equations of the two tangents at these points.
18. The slope of the tangent to the curve $y = x^4 - 1$ at the point p is 32.
    Find the coordinates of p.
19. The slope of the tangent to the curve $y = ax^2 + bx + 6$ at the point $(2, 4)$ is 3.
    Find the value of a and the value of b.
20. The slope of the tangent to the curve $y = px^2 + 1$ at the point $(1, q)$ is 6.
    Find the value of p and the value of q.
21. The curve $y = \dfrac{p + qx}{x(x + 2)}, \quad p, q \in \mathbf{R}, x \neq 0, x \neq -2$, has zero slope at the point $(1, -2)$.
    Find the value of p and the value of q.
22. A curve is given by the equation $x^2 + 4xy = 2y^2 - 8$.
    Find the coordinates of the points on the curve at which $\dfrac{dy}{dx} = 1$.

Increasing and Decreasing

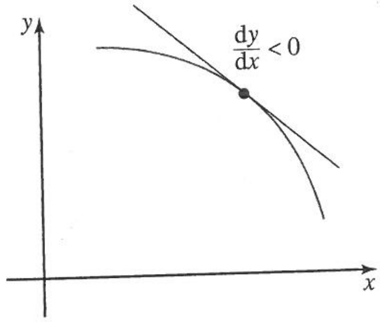
$\frac{dy}{dx}$, being the slope of a tangent to a curve at any point on the curve, can be used to determine if, and where, a curve is increasing or decreasing.

Note: Graphs are read from left to right.

Where a curve is increasing, the tangent to the curve will have a positive slope. Therefore, where a curve is increasing, $\frac{dy}{dx}$ will be positive.



Where a curve is decreasing, the tangent to the curve will have a negative slope. Therefore, where a curve is decreasing, $\frac{dy}{dx}$ will be negative



Example

If $y = \frac{2x}{1-x}$, show that $\frac{dy}{dx} > 0$ for all $x \neq 1$.

Solution:

$$y = \frac{2x}{1-x}$$

$$\frac{dy}{dx} = \frac{(1-x)(2)-(2x)(-1)}{(1-x)^2} \qquad \text{(quotient rule)}$$

$$= \frac{2-2x+2x}{(1-x)^2}$$

$$= \frac{2}{(1-x)^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{2}{(1-x)^2}$$

$(1-x)^2 > 0$ for all $x \neq 1$, $\quad 2 > 0 \quad$ (top and bottom both positive)

$$\therefore \quad \frac{2}{(1-x)^2} > 0 \text{ for all } x \neq 1$$

$$\therefore \quad \frac{dy}{dx} > 0 \text{ for all } x \neq 1.$$

Note: (any real number)^2 will always be a positive number unless the number is zero
$\therefore (1-x)^2$ must always be positive, unless $x = 1$, which gives $0^2 = 0$.

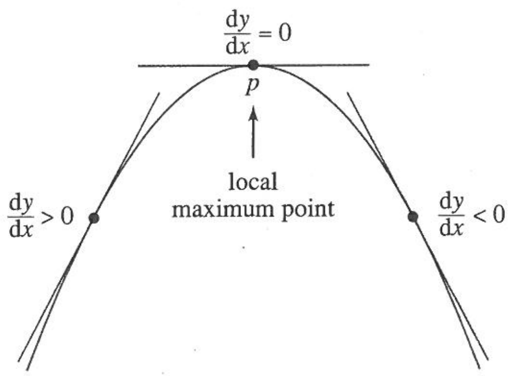
Exercise 13.2

1. Let $f(x) = x^2 - 2x - 8$. Find the values of x for which $f(x)$ is **(i)** decreasing **(ii)** increasing.
2. Let $f(x) = x^3 + 4x + 2$. Show that $\frac{dy}{dx} > 0$ for all $x \in \mathbf{R}$.
3. Let $y = \frac{x+2}{x-1}$. Show that $\frac{dy}{dx} < 0$ for all $x \in \mathbf{R}, x \neq 1$.
4. Let $y = 10 - 3x + 3x^2 - x^3$. Show that $\frac{dy}{dx} < 0$ for all $x \in \mathbf{R}$.
5. Let $f(x) = x^3 - 3x^2 - 9x + 2$. Find the values of x for which $f(x) < 0$.
6. Let $f(x) = \frac{x^2+3}{x+1}$. Find the values of x for which $f'(x) > 0$.
7. Let $f(x) = x - \sin x$. Show that $f'(x) > 0$ for $0 < x < \frac{\pi}{2}$.
8. An artificial ski-slope is described by the function $h = 2 - 8s - 4s^2 - \frac{2}{3}s^3$, where s is the horizontal distance and h is the height of the slope. Show that the slope is all downhill.
9. Let $f(x) = x \ln x$, $\quad x > 0$. Find the values of x for which $f'(x) > 0$.
10. $f(x) = \frac{\sin x + \cos x}{\sin x - \cos x}$. Show that $f(x)$ is decreasing for all $x \in \mathbf{R}$, $\tan x \neq 1$.
11. Prove that the curve $y = \frac{px+q}{rx+s}$, $\quad x \neq -\frac{s}{r}$, is increasing for all x, as long as $ps - qr > 0$.

Local Maximum Point, Local Minimum Point and Point of Inflection

Local maximum point

To the left of *p*	At *p*	To the right of *p*
$\frac{dy}{dx} > 0$	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} < 0$



As the curve passes through the point p,

$\frac{dy}{dx}$ changes from positive to negative,

i.e. $\frac{dy}{dx}$ is decreasing.

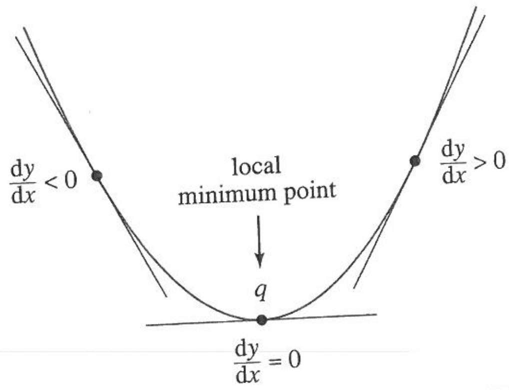
Thus, the rate of change of $\frac{dy}{dx}$ is negative,

i.e. $\frac{d^2y}{dx^2} < 0$ for a maximum point.

For a local maximum point:
$$\frac{dy}{dx} = 0 \quad \text{and} \quad \frac{d^2y}{dx^2} < 0$$

Local minimum point

To the left of q	At q	To the right of q
$\frac{dy}{dx} < 0$	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} > 0$



As the curve passes through the point q,

$\frac{dy}{dx}$ changes from negative to positive,

i.e. $\frac{dy}{dx}$ is increasing. Thus, the rate of change of $\frac{dy}{dx}$ is positive,

i.e. $\frac{d^2y}{dx^2} > 0$ for a minimum point.

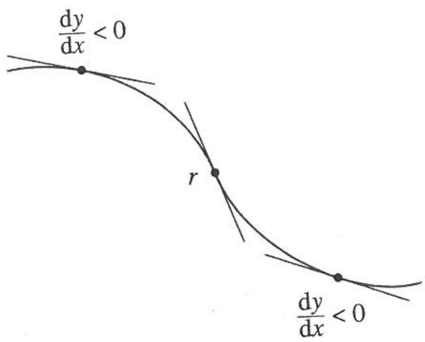
For a local minimum point:

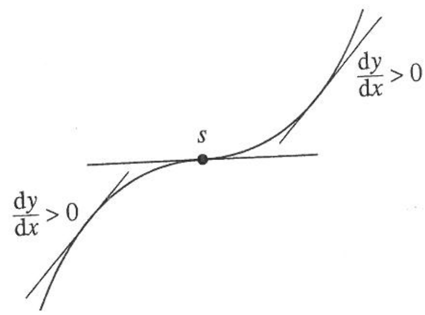
$$\frac{dy}{dx} = 0 \quad \text{and} \quad \frac{d^2y}{dx^2} > 0$$

Note: Local maximum points or local minimum points are also called '**turning points**'. They are called 'local maximum points' or 'local minimum points' as the terms 'maximum' and 'minimum' values apply only in the vicinity of (close to) the turning points, and not to the values of y in general.

Point of Inflection

This is a point at which the curvature of a curve changes. In other words, at a point of inflection, a curve stops bending in one direction and starts bending the other way. At a point of inflection, the tangent to the curve cuts the curve at that point.





The points r and s are points of inflection.

Note: The point s is called a '**horizontal point of inflection**' or '**saddle point**'.

The slope of the tangent, $\frac{dy}{dx}$, does **not** change sign as a curve passes through a point of inflection.

For a point of inflection:

$$\frac{d^2y}{dx^2} = 0 \quad \text{and} \quad \frac{d^3y}{dx^3} \neq 0$$

Note: If $\frac{d^3y}{dx^3} = 0$, it will be necessary to consider the sign of $\frac{d^2y}{dx^2}$ on either side of the point of inflection. $\frac{d^2y}{dx^2}$ **changes** sign before and after a point of inflection.

Alternatively, $\frac{dy}{dx}$ does **not** change sign on either side of a point of inflection.

Summary of conditions for a function $y = f(x)$:

1.	Increasing	$\frac{dy}{dx} > 0$
2.	Decreasing	$\frac{dy}{dx} < 0$
3.	Maximum point	$\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$
4.	Minimum point	$\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$
5.	Point of inflection	$\frac{d^2y}{dx^2} = 0$ and $\frac{d^3y}{dx^3} \neq 0$

Note: Points on a curve where $\frac{dy}{dx} = 0$ are called '**stationary points**'. At a stationary point, the tangent to the curve is horizontal. Local maximum turning points, local minimum turning points and horizontal points of inflection (saddle points) are stationary points.

Example

Find the coordinates of the local maximum point, the local minimum point and the point of inflection of the curve $y = x^3 - 3x^2 + 5$.
Draw a rough graph of the curve $y = x^3 - 3x^2 + 5$.

Solution:

$$y = x^3 - 3x^2 + 5$$
$$\frac{dy}{dx} = 3x^2 - 6x$$
$$\frac{d^2y}{dx^2} = 6x - 6$$

For a maximum or a minimum:

$$\frac{dy}{dx} = 0$$
$$\therefore \quad 3x^2 - 6x = 0$$
$$x^2 - 2x = 0$$
$$x(x - 2) = 0$$
$$x = 0 \quad \text{or} \quad x = 2$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=0} = 6(0) - 6 = -6 < 0$$

$\therefore$ local maximum at $x = 0$

$x = 0; \quad y = (0)^3 - 3(0)^2 + 5 = 5$

$\therefore$ local maximum point is $(0, 5)$

$$\left.\frac{d^2y}{dx^2}\right|_{x=2} = 6(2) - 6 = 6 > 0$$

$\therefore$ local minimum at $x = 2$

$x = 2; \quad y = (2)^3 - 3(2)^2 + 5 = 1$

$\therefore$ local minimum point is $(2, 1)$