Limits of Sequences

Consider the sequence given by $u_n = \frac{4n-1}{3n+2}$.

The terms are $\frac{3}{5}, \frac{7}{8}, \frac{11}{11}, \frac{15}{14}, \frac{19}{17}, \frac{23}{20}, \frac{27}{23}, \frac{31}{26}, \ldots$.

$u_{20} = \frac{79}{62}$ $\quad\quad$ $u_{100} = \frac{399}{302}$ $\quad\quad$ $u_{1000} = \frac{3999}{3002}$

As n gets larger, the sequence approaches $\frac{4}{3}$.

We say that the sequence has a limit of $\frac{4}{3}$ and is **convergent**.

Mathematically this is written:

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{4n-1}{3n+2} = \frac{4}{3}$$

Notes: Not all sequences have limits. A sequence which does not have a limit is said to be **divergent**.
'lim' is the abbreviation for limit.

The phrase 'n tends to infinity', written '$n \to \infty$', means that n can be made as large as we please.

Let us consider the value of the expression $\frac{1}{n}$ as $n \to \infty$.

n	$\frac{1}{n}$
10	0.1
100	0.01
1000	0.001
1,000,000	0.000001
1,000,000,000	0.000000001

The table indicates that as $n \to \infty, \frac{1}{n} \to 0$.

This is written:

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

This limit can be extended:

$$\lim_{n\to\infty} \frac{c}{n^p} = 0, \text{ for } p > 0,\ c \text{ a constant.}$$

To evaluate the limit, $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ do the following:

Divide the top and bottom by the dominant term and use the limit above.

The dominant term is the largest term as $n \to$ infinity.
In this section the dominant term is the highest power of n.

Example

Find $\lim_{n\to\infty} u_n$ if: (i) $u_n = \dfrac{n^2+5}{2n^2-3}$ (ii) $u_n = \dfrac{1}{4} - \dfrac{2}{n+5}$.

Solution:

(i)

$$u_n = \frac{n^2+5}{2n^2-3}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n^2+5}{2n^2-3}$$

$$= \lim_{n\to\infty} \frac{1+\frac{5}{n^2}}{2-\frac{3}{n^2}}$$

(divide top and bottom by n^2, the dominant term)

$$= \frac{1+0}{2-0} = \frac{1}{2}$$

(ii)

$$u_n = \frac{1}{4} - \frac{2}{n+5}$$

$$\lim_{n\to\infty} u_n = \lim\left(\frac{1}{4} - \frac{2}{n+5}\right)$$

$$= \lim\left(\frac{1}{4} - \frac{\frac{2}{n}}{1+\frac{5}{n}}\right)$$

(divide top and bottom by n, the dominant term)

$$= \frac{1}{4} - \frac{0}{1+0} = \frac{1}{4} - 0 = \frac{1}{4}$$

Sometimes we have to deal with limits that involve square roots.

For example: If $u_n = \dfrac{\sqrt{2n^2+3}}{n}$, evaluate $\lim_{n\to\infty} u_n$.

In these cases, we write the total expression under one square root and then take the square root outside the limit.

Mathematically speaking: $\lim_{n\to\infty} \sqrt{u_n} = \sqrt{\lim_{n\to\infty} u_n}$

Taking the square root outside makes no difference, as $n\to\infty$.

Example

$u_n = \dfrac{\sqrt{n^2-3n+2}}{4n+1}$. Find $\lim_{n\to\infty} u_n$.

Solution:

$$u_n = \frac{\sqrt{n^2-3n+2}}{4n+1} = \frac{\sqrt{n^2-3n+2}}{\sqrt{(4n+1)^2}} = \frac{\sqrt{n^2-3n+2}}{\sqrt{16n^2+8n+1}} = \sqrt{\frac{n^2-3n+2}{16n^2+8n+1}}$$

$(\sqrt{(4n+1)^2} = 4n+1)$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{\sqrt{n^2-3n+2}}{4n+1}$$

$$= \lim_{n\to\infty} \sqrt{\frac{n^2-3n+2}{16n^2+8n+1}}$$

$$= \sqrt{\lim_{n\to\infty} \frac{n^2-3n+2}{16n^2+8n+1}}$$

(take $\sqrt{\ }$ outside; makes no difference as $n\to\infty$)

$$= \sqrt{\lim_{n\to\infty} \frac{1-\frac{3}{n}+\frac{2}{n^2}}{16+\frac{8}{n}+\frac{1}{n^2}}} \qquad \left(\begin{array}{c}\text{divide top and bottom by } n^2,\\ \text{the dominant term}\end{array}\right)$$

$$= \sqrt{\frac{1-0+0}{16+0+0}} = \sqrt{\frac{1}{16}} = \frac{\sqrt{1}}{\sqrt{16}} = \frac{1}{4}$$

Exercise 7.7

In each of the following, find $\lim_{n\to\infty} u_n$ if:

1. $u_n = \frac{2n-1}{n+1}$
2. $u_n = \frac{4n+1}{3n-2}$
3. $u_n = \frac{2n+1}{3n+5}$
4. $u_n = \frac{4n^2+2n}{5n^2-3}$
5. $u_n = \frac{3n^2-4n}{7n^2+2n}$
6. $u_n = \frac{4}{3n+1}$
7. $u_n = \frac{n}{n+2}$
8. $u_n = \frac{1}{5} - \frac{1}{n+1}$
9. $u_n = \frac{3}{4} - \frac{5}{n+2}$
10. $u_n = \frac{1}{2} - \frac{1}{2(2n+1)}$
11. $u_n = \frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}$
12. $u_n = \frac{4}{5} - \frac{1}{n} - \frac{1}{n+3}$
13. $u_n = \sqrt{\frac{25n+2}{n-3}}$
14. $u_n = \frac{\sqrt{n-1}}{\sqrt{9n+4}}$
15. $u_n = \frac{\sqrt{4n^2-1}}{n+3}$
16. $u_n = \frac{1}{\sqrt{n^2+2}}$
17. $u_n = \frac{\sqrt{2n^2+3}}{n}$
18. $u_n = \frac{\sqrt{3n^2-1}}{2n+3}$
19. **(i)** $1+3+5+7+\cdots$ is an arithmetic series.
    Find, in terms of n: **(a)** u_n **(b)** S_n.
    (ii) Evaluate: $\lim_{n\to\infty} \frac{\sqrt{1+3+5+7+\cdots+u_n}}{2n}$.

Series of the Form $\sum_{r=1}^{n} \frac{1}{r(r+2)}$ and $\sum_{r=1}^{n} \frac{1}{(r+1)(r+3)}$

Infinite Series

Consider the infinite series $u_1+u_2+u_3+\cdots$.

This series can be written using the sigma notation:

$$\sum_{n=1}^{\infty} u_n = u_1+u_2+u_3+\cdots$$

The sum to infinity of a series is denoted by $\lim_{n\to\infty} S_n$, or simply S_∞.

Thus, we have the following:

$$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \cdots = \lim_{n\to\infty} S_n$$

In this section we will show how to find a concise formula for S_n, the sum to n terms and, hence, evaluate $\lim_{n\to\infty} S_n$, the sum to infinity, of series of the forms

$$\sum_{r=1}^{n} \frac{1}{r(r+2)} \quad \text{and} \quad \sum_{r=1}^{n} \frac{1}{(r+1)(r+3)},$$ which are neither arithmetic nor geometric.

As with infinite geometric series:

if $\lim_{n\to\infty} S_n$ exists, the series is said to be **convergent**;

if $\lim_{n\to\infty} S_n$ does **not** exist, the series is said to be **divergent**.

Note: On this part of our course the series will be confined to those for which $\lim_{n\to\infty} S_n$ exists (i.e. the sum of the series can be found).

The sum of an infinite convergent series is found with the following steps:

1. Find a concise expression for S_n. **2.** Evaluate $\lim_{n\to\infty} S_n$.

Partial Fractions and Telescoping the Series

By algebraic addition, $\frac{1}{n+1} - \frac{1}{n+2} = \frac{1}{(n+1)(n+2)}$.

The reverse process of showing that $\frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}$

is called resolving into **partial fractions**.

Partial fractions enable us to find a concise expression for S_n for a series of the form $\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)}$ or $\sum_{r=1}^{n} \frac{1}{r(r+2)}$, using the following steps:

1. Express the nth term in the form $u_r = v_r - v_{r+1}$ (or similar).
2. List the terms vertically.
3. Add the terms (most will cancel).

This will leave us with a concise expression for S_n (as most of the terms will cancel).
The process which enables us to find S_n of a series through cancellation is called **telescoping** a series (the series 'folds in' on itself).

Example

(i) If $u_k = \dfrac{1}{(k+2)(k+3)} = \dfrac{A}{k+2} + \dfrac{B}{k+3}$, for all $k \in \mathbf{R}$,

find the value of A and the value of B, $A, B \in \mathbf{R}$.

(ii) **(a)** Find, in terms of n: $\sum_{k=1}^{n} u_k$. **(b)** Evaluate: $\sum_{k=1}^{100} u_k$.

(iii) Evaluate: $\sum_{k=1}^{\infty} u_k$.

(iv) Find the value of n such that $\sum_{k=1}^{n} u_k = \frac{24}{25}\sum_{k=1}^{\infty} u_k$.

Solution:

(i) $$\frac{1}{(k+2)(k+3)} = \frac{A}{(k+2)} + \frac{B}{(k+3)}$$

$$1 = A(k+3) + B(k+2) \qquad \text{[multiply both sides by } (k+2)(k+3)]$$

What we do next is choose two values for k and substitute these into the equation. The two most suitable values are ones that make the coefficient of $B = 0$ and the coefficient of $A = 0$.

The values of k where $(k+2)$ and $(k+3)$ are zero are used as follows:

$$1 = A(k+3) + B(k+2)$$

Let $k=-2$:	Let $k=-3$:
$1 = A(-2+3) + B(-2+2)$	$1 = A(-3+3) + B(-3+2)$
$1 = A(1) + B(0)$	$1 = A(0) + B(-1)$
$1 = A$	$1 = -B$
	$-1 = B$

$$\therefore \quad \frac{1}{(k+2)(k+3)} = \frac{1}{k+2} - \frac{1}{k+3}$$

Note: There are two other methods for finding the value of A and the value of B.

1. Substitute any two values for k to obtain two equations in A and B. Solve these simultaneous equations. (If the question says $k \in \mathbf{N}$, then you can use only positive whole numbers.)
2. Remove the brackets and equate the coefficients of like terms to obtain two equations in A and B. Solve these simultaneous equations.

(ii) (a) $\displaystyle\sum_{k=1}^{n} u_k = \sum_{k=1}^{n} \frac{1}{(k+2)(k+3)} = \sum_{k=1}^{n}\left(\frac{1}{k+2} - \frac{1}{k+3}\right) = S_n$

What we do next is telescope the series by listing the terms vertically and cancelling terms which are the same but of opposite signs. It is good practice to write down enough terms to see the terms that cancel. It always happens that the same number of terms remain at the top and the bottom.

$$
\begin{array}{rrcrcl}
 & u_k & = & \dfrac{1}{k+2} & - & \dfrac{1}{k+3} \\
\text{(one term remaining)} \longrightarrow & u_1 & = & \dfrac{1}{3} & - & \cancel{\dfrac{1}{4}} \\
 & u_2 & = & \cancel{\dfrac{1}{4}} & - & \cancel{\dfrac{1}{5}} \\
 & u_3 & = & \cancel{\dfrac{1}{5}} & - & \cancel{\dfrac{1}{6}} \\
 & \vdots & & \vdots & & \vdots \\
 & \vdots & & \vdots & & \vdots \\
 & \vdots & & \vdots & & \vdots \\
 & u_{n-2} & = & \cancel{\dfrac{1}{n}} & - & \cancel{\dfrac{1}{n+1}} \\
 & u_{n-1} & = & \cancel{\dfrac{1}{n+1}} & - & \cancel{\dfrac{1}{n+2}} \\
 & u_n & = & \cancel{\dfrac{1}{n+2}} & - & \dfrac{1}{n+3} \longleftarrow \text{(one term remaining)} \\
\hline
 & S_n & = & \dfrac{1}{3} & - & \dfrac{1}{n+3} \quad \text{(adding)}
\end{array}
$$

(b) $\displaystyle\sum_{k=1}^{100} u_k = S_{100}$

$$S_n = \frac{1}{3} - \frac{1}{n+3}$$

$$S_{100} = \frac{1}{3} - \frac{1}{100+3} \qquad \text{[put in } n = 100\text{]}$$

$$= \frac{1}{3} - \frac{1}{103} = \frac{100}{309}$$

10. **(i)** Explain why $\sqrt{a}\sqrt{a+1}=\sqrt{a^2+a}$.

(ii) Show that $\dfrac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k^2+k}}=\dfrac{1}{\sqrt{k}}-\dfrac{1}{\sqrt{k+1}}$.

(iii) If $u_k=\dfrac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k^2+k}}$, express, in terms of n: $\sum\limits_{k=1}^{n} u_k$.

(iv) Evaluate: **(a)** $\sum\limits_{k=1}^{80} u_n$ **(b)** $\sum\limits_{k=1}^{\infty} u_n$.

11. If $u_r=\ln\left(\dfrac{r+1}{r}\right)$, express, in terms of n: $\sum\limits_{r=1}^{n} u_r$.

(Hint: $\ln\dfrac{a}{b}=\ln a-\ln b$, and telescope the series.)

Evaluate $\sum\limits_{r=1}^{53} u_r$, correct to two significant figures.

12. Show that $\dfrac{k}{(k+1)!}=\dfrac{1}{k!}-\dfrac{1}{(k+1)!}$.

If $u_k=\dfrac{k}{(k+1)!}$, express in terms of n: $\sum\limits_{k=1}^{n} u_k$.

Series of Powers of Natural Numbers

There are three series on our course involving series of powers of natural numbers.

1. $\sum\limits_{r=1}^{n} k=k+k+k+\cdots+k=nk$

2. $\sum\limits_{r=1}^{n} r=1+2+3+\cdots+n=\dfrac{n}{2}(n+1)$

3. $\sum\limits_{r=1}^{n} r^2=1^2+2^2+3^2+\cdots+n^2=\dfrac{n}{6}(n+1)(2n+1)$

We use these to find an expression for S_n if $u_n=an^2+bn+c$, where $a, b, c \in \mathbf{R}$.

Notes: **(i)** $\sum\limits_{r=1}^{n} 1=\underbrace{(1+1+1+\cdots+1)}_{\longleftarrow\ n \text{ times}\ \longrightarrow}=n$ (since a 1 is required for each $r=1, 2, 3, \ldots n$).

(ii) Each of these formulae can be proved using induction.
(1 and 2 are arithmetic series and can be proved using the sum to n terms, $S_n=\dfrac{n}{2}[2a+(n-1)d]$.)

(iii) $\sum\limits_{r=1}^{n}(u_r+v_r)=\sum\limits_{r=1}^{n}u_r+\sum\limits_{r=1}^{n}v_r$ and $\sum\limits_{r=1}^{n}ku_r=k\sum\limits_{r=1}^{n}u_r$.

Example

(i) Write $\dfrac{n^3+1}{n+1}$ in the form an^2+bn+c, where $a, b, c \in \mathbf{R}$.

(ii) Hence, evaluate $\displaystyle\sum_{n=1}^{24}\frac{n^3+1}{n+1}$.

Solution:

(i) Method 1 (using factors):

$$\frac{n^3+1}{n+1}$$

$$= \frac{(n)^3+(1)^3}{n+1} \qquad \left(\begin{array}{c}\text{sum of two}\\ \text{cubes on top}\end{array}\right)$$

$$= \frac{(n+1)(n^2-n+1)}{(n+1)}$$

$$= n^2-n+1$$

(i) Method 2 (using long division):

$$\begin{array}{r|l} & n^2-n+1 \\ \hline n+1 & n^3+0n^2+0n+1 \\ & \underline{n^3+n^2} \\ & \quad -n^2+0n \\ & \quad \underline{-n^2-n} \\ & \qquad\quad n+1 \\ & \qquad\quad \underline{n+1} \\ & \qquad\quad 0+0 \end{array}$$

Thus, $\dfrac{n^3+1}{n+1} = n^2-n+1$.

(ii) $$\sum_{n=1}^{24}\frac{n^3+1}{n+1}$$

$$= \sum_{n=1}^{24}(n^2-n+1)$$

$$= \sum_{n=1}^{24}n^2 - \sum_{n=1}^{24}n + \sum_{n=1}^{24}1$$

$$\left(\sum_{r=1}^{n}r^2 = \frac{n}{6}(n+1)(2n+1); \qquad \sum_{r=1}^{n}r = \frac{n}{2}(n+1); \qquad \sum_{r=1}^{n}1 = n\right)$$

$$= \frac{24}{6}(24+1)(48+1) - \frac{24}{2}(24+1) + 24 \qquad \text{(put in } n=24)$$

$$= 4(25)(49) - 12(25) + 24$$

$$= 4{,}900 - 300 + 24$$

$$= 4{,}624$$

Example

$$\sum_{r=1}^{n}(3r-1)(2r+4) = 2.6 + 5.8 + 8.10 + \cdots + (3n-1)(2n+4)$$

Find, in terms of n, an expression for $\sum_{r=1}^{n}(3r-1)(2r+4)$.

Solution:

$$u_r = (3r-1)(2r+4) = 6r^2 + 10r - 4$$

$$\therefore \quad \sum_{r=1}^{n}(3r-1)(2r+4)$$

$$= \sum_{r=1}^{n}(6r^2 + 10r - 4)$$

$$= \sum_{r=1}^{n}6r^2 + \sum_{r=1}^{n}10r - \sum_{r=1}^{n}4$$

$$= 6\sum_{r=1}^{n}r^2 + 10\sum_{r=1}^{n}r - 4\sum_{r=1}^{n}1$$

$$= 6\left[\frac{n}{6}(n+1)(2n+1)\right] + 10\left[\frac{n}{2}(n+1)\right] - 4[n]$$

$$= n(n+1)(2n+1) + 5n(n+1) - 4n$$

$$= n[(n+1)(2n+1) + 5(n+1) - 4] \qquad \text{(factor out } n\text{)}$$

$$= n(2n^2 + 3n + 1 + 5n + 5 - 4)$$

$$= n(2n^2 + 8n + 2)$$

$$= 2n(n^2 + 4n + 1)$$

Some sequences are a combination of powers of natural numbers and a geometric sequence. For example, $u_n = 2n^2 + 3n + 5^n$.
These sequences can be split and summed separately.

Example

Find, in terms of n, $\sum_{r=1}^{n}(2r - 1 + 3^r)$. Hence, evaluate $\sum_{n=1}^{8}(2n - 1 + 3^n)$

Solution:

u_n is made up of a 'sum of powers of natural numbers' part, $2r-1$, and a 'geometric' part, 3^r.
We sum these parts separately and combine the results.

Sum of the powers of natural numbers part:

$$\sum_{r=1}^{n}(2r-1)$$
$$=\sum_{r=1}^{n}2r-\sum_{r=1}^{n}1$$
$$=2\sum_{r=1}^{n}r-\sum_{r=1}^{n}1$$
$$=2\,.\,\frac{n}{2}(n+1)-n$$
$$=n(n+1)-n$$
$$=n^2+n-n$$
$$=n^2$$

Sum of the geometric part:

$$\sum_{r=1}^{n}3^r$$
$$=\frac{a(r^n-1)}{r-1}$$
$$=\frac{3(3^n-1)}{3-1}$$
$$=\frac{3(3^n-1)}{2}$$

Thus, $\displaystyle\sum_{r=1}^{n}(2r-1+3^r)=n^2+\frac{3(3^n-1)}{2}$.

$$\sum_{n=1}^{8}(2n-1+3^n)=(8)^2+\frac{3(3^8-1)}{2} \qquad \text{(put in } n=8\text{)}$$
$$=64+\frac{3(6{,}561-1)}{2}$$
$$=64+9{,}840=9{,}904$$

Note: $\displaystyle\sum_{r=1}^{n}(2r-1)$ is the arithmetic series $1+3+5+\cdots+(2r-1)$.

This part could have been summed using the formula $S_n=\dfrac{n}{2}[2a+(n-1)d]$.

Exercise 7.9

Show that:

1. $1+3+5+7+\cdots+(2n-1)=n^2$
2. $1+5+9+13+\cdots+(4n-3)=2n^2-n$
3. $2\,.\,2+4\,.\,5+6\,.\,8+\cdots+2n(3n-1)=2n^2(n+1)$
4. $2\,.\,4+5\,.\,6+8\,.\,8+\cdots+(3n-1)(2n+2)=n(2n^2+5n+1)$
5. $3\,.\,7+5\,.\,13+7\,.\,19+\cdots+(2n+1)(6n+1)=n(4n^2+10n+7)$
6. Evaluate: **(i)** $\displaystyle\sum_{n=1}^{20}(2n+3)$ **(ii)** $\displaystyle\sum_{n=1}^{20}n(2n+1)$.
7. $\displaystyle\sum_{r=1}^{n}2r(3r-1)=an^2(n+b)$. Find the value of a and the value of b.

   Hence, or otherwise, evaluate $\displaystyle\sum_{n=1}^{9}2n(3n-1)$.

8. (i) Express $(2r-1)^2$ in the form ar^2+br+c, where $a, b, c \in \mathbf{Z}$.

(ii) Show that $\sum_{r=1}^{n} 3(2r-1)^2 = n(2n-1)(2n+1)$.

(iii) Hence, evaluate $\sum_{n=1}^{10} 3(2n-1)^2$.

9. (i) Express $\dfrac{n^2+6n+5}{n+1}$ in the form $an+b$, where $a, b \in \mathbf{R}$.

(ii) Hence, evaluate $\sum_{n=1}^{100} \dfrac{n^2+6n+5}{n+1}$.

10. (i) Express $\dfrac{n^3+64}{n+4}$ in the form an^2+bn+c, where $a, b, c \in \mathbf{R}$.

(ii) Hence, evaluate $\sum_{n=1}^{21} \dfrac{n^3+64}{n+4}$.

11. Show that $\sum_{r=1}^{n} (3r^2+r-2) = n(n^2+2n-1)$. Hence, evaluate $\sum_{n=1}^{20} (3n^2+n-2)$.

Evaluate each of the following:

12. $\sum_{n=1}^{6} (2+3^n)$ 13. $\sum_{n=1}^{12} (5n+3+2^n)$ 14. $\sum_{n=1}^{10} (2^n-n^2+1)$

15. (i) Express $\sum_{r=1}^{n} (r+1)(2r+1)$ in the form $\dfrac{n}{6}(an^2+bn+c)$, where $a, b, c \in \mathbf{N}$.

(ii) Hence, evaluate: $2.3+3.5+4.7+\cdots+11.21$.

16. For a certain sequence, $u_n = an^2+bn+c$, where $a, b, c \in \mathbf{Z}$.
If $u_1=4$, $u_2=15$ and $u_3=2u_2$, find the values of a, b and c.
Hence, evaluate $\sum_{n=1}^{12} an^2+bn+c$.

17. $\sum_{r=1}^{n} r = 1+2+3+\cdots+n$. Show that $\sum_{r=1}^{n} r = \dfrac{n}{2}(n+1)$.
Evaluate: $1+(1+2)+(1+2+3)+\cdots+(1+2+3+\cdots+20)$.

CHAPTER 12

DIFFERENTIAL CALCULUS 2 DIFFERENTIATION BY RULE

Differentiation by Rule

Differentiation from first principles can become tedious and difficult. Fortunately, it is not always necessary to use first principles. There are a few rules (which can be derived from first principles) which enable us to write down the derivative of a function quite easily.

Rule 1: General Rule

If:

$$y = x^n \quad \text{then} \quad \frac{dy}{dx} = nx^{n-1}$$
$$y = ax^n \quad \text{then} \quad \frac{dy}{dx} = nax^{n-1}$$

In words:

Multiply by the power and reduce the power by 1.

Example

Differentiate with respect to x:

(i) $y = x^5$ **(ii)** $y = -3x^2$ **(iii)** $y = 5x$ **(iv)** $y = \frac{8}{x^2}$

(v) $y = 6\sqrt{x}$ **(vi)** $y = \frac{2}{\sqrt{x}}$ **(vii)** $y = \frac{6}{x^{1/3}}$ **(viii)** $y = 7$

Solution:

(i) $y = x^5$ $\qquad \frac{dy}{dx} = 5x^{5-1} = 5x^4$

(ii) $y = -3x^2$ $\qquad \frac{dy}{dx} = 2 \times -3x^{2-1} = -6x$

(iii) $y = 5x = 5x^1$ $\qquad \frac{dy}{dx} = 1 \times 5x^{1-1} = 5x^0 = 5 \quad (x^0 = 1)$

(iv) $y = \frac{8}{x^2} = 8x^{-2}$ $\qquad \frac{dy}{dx} = -2 \times 8x^{-2-1} = -16x^{-3} = -\frac{16}{x^3}$

(v) $y = 6\sqrt{x} = 6x^{1/2}$ $\qquad$ $\dfrac{dy}{dx} = \frac{1}{2} \times 6x^{1/2-1} = 3x^{-1/2} = \dfrac{3}{x^{1/2}} = \dfrac{3}{\sqrt{x}}$

(vi) $y = \dfrac{2}{\sqrt{x}} = 2x^{-1/2}$ $\qquad$ $\dfrac{dy}{dx} = -\frac{1}{2} \times 2x^{-1/2-1} = -1x^{-3/2} = -\dfrac{1}{x^{3/2}}$

(vii) $y = \dfrac{6}{x^{1/3}} = 6x^{-1/3}$ $\qquad$ $\dfrac{dy}{dx} = -\frac{1}{3} \times 6x^{-1/3-1} = -2x^{-4/3} = -\dfrac{2}{x^{4/3}}$

(viii) $y = 7 = 7x^0$ $\qquad$ $\dfrac{dy}{dx} = 0 \times 7x^{0-1} = 0$

Part (viii) leads to the rule:

> The derivative of a constant = 0.

Note: The line $y = 7$ is a horizontal line. Its slope is 0.
Therefore its derivative (also its slope) equals 0.
In other words, the derivative of a constant always equals zero.

Sum or Difference
If the expression to be differentiated contains more than one term, just differentiate, separately, eac
term in the expression.

Example

Find $f'(x)$ for each of the following:

(i) $f(x) = x + \dfrac{1}{x^2}$

(ii) $f(x) = \dfrac{2}{\sqrt{x}} - \dfrac{1}{x^4} + 5$

Solution:

(i)
$$f(x) = x + \frac{1}{x^2}$$
$$f(x) = x + x^{-2}$$
$$f'(x) = 1 - 2x^{-3}$$
$$f'(x) = 1 - \frac{2}{x^3}$$

(ii)
$$f(x) = \frac{2}{\sqrt{x}} - \frac{1}{x^4} + 5$$
$$f(x) = 2x^{-1/2} - x^{-4} + 5$$
$$f'(x) = -x^{-3/2} + 4x^{-5}$$
$$f'(x) = -\frac{1}{x^{3/2}} + \frac{4}{x^5}$$

Evaluating Derivatives

Often we have to evaluate a derivative for a particular value.

Example

(i) If $s = 3t^2 + 5t - 7$, find the value of $\dfrac{\mathrm{d}s}{\mathrm{d}t}$ when $t = 2$.

(ii) If $f(x) = \sqrt{x} + 3x$, evaluate $f'(4)$.

Solution:

(i)

$$s = 3t^2 + 5t - 7$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = 6t + 5$$

$$\left.\frac{\mathrm{d}s}{\mathrm{d}t}\right|_{t=2} = 6(2) + 5$$

$$= 12 + 5 = 17$$

(ii)

$$f(x) = \sqrt{x} + 3x$$

$$f(x) = x^{1/2} + 3x$$

$$f'(x) = \tfrac{1}{2}x^{-1/2} + 3$$

$$= \frac{1}{2\sqrt{x}} + 3$$

$$f'(4) = \frac{1}{2\sqrt{4}} + 3$$

$$= \frac{1}{4} + 3 = 3\tfrac{1}{4}$$

$\dfrac{\mathrm{d}s}{\mathrm{d}t}$ is the derivative of s with respect to t. $\dfrac{\mathrm{d}A}{\mathrm{d}r}$ is the derivative of A with respect to r.

Second Derivatives

The derivative of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, that is $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)$, is denoted by $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ and is called the

'second derivative of y with respect to x'.

$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is pronounced 'dee two y, dee x squared'.

The derivative of $f'(x)$ is denoted by $f''(x)$ and is called the

'second derivative of $f(x)$ with respect to x'.

Example

(i) If $f(x) = x + \frac{1}{x}$, find $f''(x)$ and $f''(2)$.

(ii) If $h = 10 + 30t^2 - 4t^3$, evaluate $\frac{d^2h}{dt^2}$ when $t = 3$.

Solution:

(i)
$$f(x) = x + \frac{1}{x}$$
$$f(x) = x + x^{-1}$$
$$f'(x) = 1 - x^{-2}$$
$$f''(x) = 2x^{-3} = \frac{2}{x^3}$$
$$\therefore\ f''(2) = \frac{2}{2^3} = \frac{2}{8} = \frac{1}{4}$$

(ii)
$$h = 10 + 30t^2 - 4t^3$$
$$\frac{dh}{dt} = 60t - 12t^2$$
$$\frac{d^2h}{dt^2} = 60 - 24t$$
$$\left.\frac{d^2h}{dt^2}\right|_{t=3} = 60 - 24(3) = -12$$

Example

If $y = x^4$, show that $\frac{4y}{3}\left(\frac{d^2y}{dx^2}\right) - \left(\frac{dy}{dx}\right)^2 = 0$.

Solution:

$$y = x^4$$
$$\frac{dy}{dx} = 4x^3$$
$$\frac{d^2y}{dx^2} = 12x^2$$

$$\frac{4y}{3}\left(\frac{d^2y}{dx^2}\right) - \left(\frac{dy}{dx}\right)^2$$
$$= \frac{4x^4}{3}(12x^2) - (4x^3)^2$$
$$= 16x^6 - 16x^6$$
$$= 0$$

Note: $\left(\frac{dy}{dx}\right)^2 \neq \frac{d^2y}{dx^2}$

Exercise 12.1

Differentiate each of the following with respect to x:

1. x^3
2. $3x^4$
3. $-5x^2$
4. $3x$
5. $-2x$
6. 5
7. -3
8. $\frac{1}{x^2}$
9. $\frac{2}{x^3}$
10. $-\frac{2}{x^5}$

11. $6x^{1/3}$ **12.** $\dfrac{1}{x}$ **13.** $\sqrt{x}$ **14.** $\dfrac{4}{\sqrt{x}}$ **15.** $\dfrac{1}{x^{2/3}}$

16. $x^3 - 5x$ **17.** $1 - x^2$ **18.** $x^2 - \dfrac{5}{x}$ **19.** $2x^2 - \dfrac{3}{x^4}$ **20.** $\dfrac{1}{x^2} + \dfrac{1}{x}$

21. $x^4 - \dfrac{2}{x^2}$ **22.** $6\sqrt{x} - \dfrac{2}{\sqrt{x}}$ **23.** $\dfrac{3}{x} + \dfrac{2}{x^2} + \dfrac{6}{x^{1/3}}$ **24.** $\dfrac{2}{x} - \dfrac{1}{\sqrt{x}} + \dfrac{3}{x^{1/3}}$

Find $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ for each of the following:

25. $y = 4x^3 + 6x^2$ **26.** $y = x^2 - x^4$ **27.** $y = 6x^3 - 12x^2 - 8x + 4$

28. $y = \dfrac{1}{x}$ **29.** $y = x^2 - \dfrac{8}{x}$ **30.** $y = \sqrt{x}$

31. $y = \dfrac{1}{\sqrt{x}} + \sqrt{x}$ **32.** $y = 8\sqrt{x} - \dfrac{1}{x^2}$ **33.** $y = 9x^{1/3} + \dfrac{18}{x^{1/3}}$

34. If $f(x) = 3x^2 - 4x - 7$, evaluate **(i)** $f'(2)$ **(ii)** $f''(-1)$.

35. If $f(x) = -4\sqrt{x}$, evaluate $f''(9)$.

36. If $A = 3r^2 - 5r$, find the value of $\dfrac{\mathrm{d}A}{\mathrm{d}r}$ when $r = 3$.

37. If $s = 3t - 2t^2$, find the value of **(i)** $\dfrac{\mathrm{d}s}{\mathrm{d}t}$ **(ii)** $\dfrac{\mathrm{d}^2s}{\mathrm{d}t^2}$ when $t = 2$.

38. If $V = 3h - h^2 - 3h^3$, find $\dfrac{\mathrm{d}V}{\mathrm{d}h}$ when $h = 1$.

39. If $A = \pi r^2$, find $\dfrac{\mathrm{d}A}{\mathrm{d}r}$ when $\dfrac{r}{5} = 1$.

40. If $V = \frac{4}{3}\pi r^3$, find $\dfrac{\mathrm{d}V}{\mathrm{d}r}$ when $2r - 5 = 0$.

41. $f(x) = 3x^2 - 4x$. If $f'(k) = 8$, find the value of k, $k \in \mathbf{R}$.

42. $f(x) = x^3 + 1$. If $f''(a) = 18$, find the value of a, $a \in \mathbf{R}$.

43. If $y = 3x^2 + 2x$, show that $y\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3x\dfrac{\mathrm{d}y}{\mathrm{d}x} - 6x = 0$.

44. If $y = 4x^3 - 6x^2$, show that $x^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\dfrac{\mathrm{d}y}{\mathrm{d}x} - 12x = 0$.

Find the values of x for which **(i)** $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ **(ii)** $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$.

45. If $y = \dfrac{1}{x^2}$, show that $y\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + \left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - 10y^3 = 0$.

Product, Quotient and Chain Rules

Rule 2: Product Rule

Suppose u and v are functions of x.

If $y = uv$,

then $$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

In words:

First by the derivative of the second + second by the derivative of the first.

Example

If $y = (x^2 - 3x + 2)(x^2 - 2)$, find $\dfrac{dy}{dx}$.

Solution:

Let $u = x^2 - 3x + 2$ and let $v = x^2 - 2$

$\dfrac{du}{dx} = 2x - 3$ and $\dfrac{dv}{dx} = 2x$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \quad \text{(product rule)}$$
$$= (x^2 - 3x + 2)(2x) + (x^2 - 2)(2x - 3)$$
$$= 2x^3 - 6x^2 + 4x + 2x^3 - 3x^2 - 4x + 6$$
$$= 4x^3 - 9x^2 + 6$$

Rule 3: Quotient Rule

Suppose u and v are functions of x.

If $y = \dfrac{u}{v}$

then $$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

In words:

$$\frac{\text{Bottom by the derivative of the top} - \text{Top by the derivative of the bottom}}{(\text{Bottom})^2}$$

Example

If $y = \dfrac{x^2}{x-2}$, find $\dfrac{dy}{dx}$.

Solution:

Let $u = x^2$ and let $v = x - 2$

$\dfrac{du}{dx} = 2x$ and $\dfrac{dv}{dx} = 1$

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} \qquad \text{(quotient rule)}$$

$$= \frac{(x-2)(2x) - (x^2)(1)}{(x-2)^2}$$

$$= \frac{2x^2 - 4x - x^2}{(x-2)^2}$$

$$= \frac{x^2 - 4x}{(x-2)^2}$$

Note: It is usual practice to simplify the top but **not** the bottom.

Function of a function

When we write, for example, $y = (x+5)^3$, we say that y is a function of x.
If we let $u = (x+5)$, then $y = u^3$, where $u = (x+5)$.
We say that y is a function u, and u is a function of x.
The new variable, u, is the **link** between the two expressions.

Rule 4: Chain Rule

Suppose u is a function of x.

If $y = u^n$

then $\dfrac{dy}{dx} = nu^{n-1}\dfrac{du}{dx}$.

The chain rule should be done in **one** step.

Example

Find $\frac{dy}{dx}$ for each of the following:

(i) $y = (x^2 - 3x)^4$ **(ii)** $y = \frac{3}{2x+5}$

(iii) $y = \sqrt{4x - 3}$ **(iv)** $y = \left(x^2 + \frac{1}{x}\right)^3$

Solution:

(i)
$$y = (x^2 - 3x)^4$$
$$\frac{dy}{dx} = 4(x^2 - 3x)^3(2x - 3)$$
$$= (8x - 12)(x^2 - 3x)^3$$

(ii)
$$y = \frac{3}{2x+5}$$
$$y = 3(2x + 5)^{-1}$$
$$\frac{dy}{dx} = -3(2x + 5)^{-2}(2)$$
$$= \frac{-6}{(2x+5)^2}$$

(iii)
$$y = \sqrt{4x - 3}$$
$$y = (4x - 3)^{1/2}$$
$$\frac{dy}{dx} = \tfrac{1}{2}(4x - 3)^{-1/2}(4)$$
$$= \frac{2}{(4x-3)^{1/2}} = \frac{2}{\sqrt{4x-3}}$$

(iv)
$$y = \left(x^2 + \frac{1}{x}\right)^3$$
$$y = (x^2 + x^{-1})^3$$
$$\frac{dy}{dx} = 3(x^2 + x^{-1})^2(2x - x^{-2})$$
$$= 3\left(x^2 + \frac{1}{x}\right)^2\left(2x - \frac{1}{x^2}\right)$$

Often we have to deal with a combination of the product, quotient or chain rules.

Example

Find $\frac{dy}{dx}$ if **(i)** $y = x\sqrt{9 - x^2}$ **(ii)** $y = \sqrt{\frac{1-x}{1+x}}$

Solution:

(i)
$$y = x\sqrt{9 - x^2}$$
$$y = x(9 - x^2)^{1/2}$$
$$\frac{dy}{dx} = (x)\,.\,\underbrace{\tfrac{1}{2}(9 - x^2)^{-1/2}(-2x)}_{\text{(chain rule here)}} + (9 - x^2)^{1/2}(1) \qquad \text{(product rule and chain rule)}$$
$$= -x^2(9 - x^2)^{-1/2} + (9 - x^2)^{1/2}$$
$$= \frac{-x^2}{\sqrt{9-x^2}} + \sqrt{9 - x^2}$$

(ii) $y = \sqrt{\dfrac{1-x}{1+x}}$

$$y = \left(\frac{1-x}{1+x}\right)^{1/2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{-1/2}\left[\frac{(1+x)(1)-(1-x)(1)}{(1+x)^2}\right] \qquad \left(\begin{matrix}\text{chain rule followed by}\\ \text{the quotient rule}\end{matrix}\right)$$

$$= \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{1/2}\left[\frac{-1-x-1+x}{(1+x)^2}\right] \qquad \left(\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}\right)$$

$$= \frac{(1+x)^{1/2}}{2(1-x)^{1/2}} \cdot \frac{-2}{(1+x)^2}$$

$$= \frac{-1}{(1-x)^{1/2}(1+x)^{3/2}}$$

Exercise 12.2

In questions 1 to 6, use the product rule to find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$:

1. $y = (2x+3)(x-4)$
2. $y = (x+5)(x^2-3x+2)$
3. $y = (3x-4)(x^2-2x+3)$
4. $y = (x+3)(x^2-6x+8)$
5. $y = (5x^2-3x)(x^2-5x)$
6. $y = (3x^3-2x^2+4)(2x-1)$

In questions 7 to 12, use the quotient rule to find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$:

7. $y = \dfrac{3x+2}{x+1}$
8. $y = \dfrac{2x-1}{x+3}$
9. $y = \dfrac{3x-1}{x^2-2}$
10. $y = \dfrac{x^2-1}{x^2+1}$
11. $y = \dfrac{1-x}{2x-x^2}$
12. $y = \dfrac{x^2-x-6}{x^2+x-6}$

In questions 13–18, use the chain rule to find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$:

13. $y = (3x+2)^4$
14. $y = (x^2+2x)^3$
15. $y = (2x^2+1)^5$
16. $y = \sqrt{4x+2}$
17. $y = \dfrac{1}{2x-5}$
18. $y = \dfrac{1}{\sqrt{2x^2-4x}}$

Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ if:

19. $y = x^2(x+3)^4$
20. $y = 3x(x+2)^3$
21. $y = 3x^2(2x+3)^2$
22. $y = x^2\sqrt{2x+1}$
23. $y = x\sqrt{1+x^2}$
24. $y = \sqrt{\dfrac{x+1}{x}}$
25. If $f(x) = \sqrt{\dfrac{x}{x+3}}$, find the value of $f'(1)$.
26. If $f(x) = \sqrt{\dfrac{x-1}{x+1}}$, find the value of $f'(\frac{5}{4})$.

Differentiation of Trigonometric Functions

The rules for differentiating also apply to trigonometric functions.
The following are in the tables on page 41, but they are shown only for x.
The chain rule is used throughout, assuming u is a function of x.
Therefore, if you are using the tables, replace x with u and **always** multiply by $\dfrac{du}{dx}$.

Basic rule (page 41 tables)	
$f(x)$	$f'(x)$
$\cos x$	$-\sin x$
$\sin x$	$\cos x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\cot x$	$-\operatorname{cosec}^2 x$

Chain rule	
$f(u)$	$f'(u) \cdot \dfrac{du}{dx}$
$\cos u$	$-\sin u \cdot \dfrac{du}{dx}$
$\sin u$	$\cos u \cdot \dfrac{du}{dx}$
$\tan u$	$\sec^2 u \cdot \dfrac{du}{dx}$
$\sec u$	$\sec u \tan u \cdot \dfrac{du}{dx}$
$\operatorname{cosec} u$	$-\operatorname{cosec} u \cot u \cdot \dfrac{du}{dx}$
$\cot u$	$-\operatorname{cosec}^2 u \cdot \dfrac{du}{dx}$

Example

Find the derivatives of the functions:

(i) $\cos 3x$ **(ii)** $\tan^3 5x$ **(iii)** $x \sin x$ **(iv)** $\sqrt{\cos x}$

Solution:

(i)
$$y = \cos 3x$$
$$\frac{dy}{dx} = (-\sin 3x)(3)$$
$$= -3 \sin 3x$$

(ii)
$$t = \tan^3 5x$$
$$y = (\tan 5x)^3$$
$$\frac{dy}{dx} = 3(\tan 5x)^2(\sec^2 5x)(5)$$
[PTA: (power) (trig. function) (angle)]
$$= 15 \tan^2 5x \sec^2 5x$$

(iii)
$$y = x \sin x$$
(use the product rule)
$$\frac{dy}{dx} = (x)(\cos x) + (\sin x)(1)$$
$$= x \cos x + \sin x$$

(iv)
$$y = \sqrt{\cos x}$$
$$y = (\cos x)^{1/2}$$
$$\frac{dy}{dx} = \tfrac{1}{2}(\cos x)^{-1/2}(-\sin x)$$
(chain rule)
$$= \frac{-\sin x}{2\sqrt{\cos x}}$$