$$u_n \quad - \quad u_{n-1}$$
$$\downarrow \qquad \downarrow$$
$$= (3n-2)-(3n-5)$$
$$= 3n-2-3n+5$$
$$= 3 \qquad \text{(a constant)}$$
$$u_n - u_{n-1} = \text{a constant}$$

Thus, u_n is an arithmetic sequence.

$$u_n \quad - \quad u_{n+1}$$
$$\downarrow \qquad \downarrow$$
$$= (n^2-2n+5)-(n^2-4n+8)$$
$$= n^2-2n+5-n^2+4n-8$$
$$= 2n-3 \qquad \text{(not a constant)}$$
$$u_n - u_{n-1} \neq \text{a constant}$$

Thus, u_n is not an arithmetic sequence.

Alternative method for **(ii)**:

$$u_1 = (1)^2-2(1)+5 = 1-2+5 = 4$$
$$u_2 = (2)^2-2(2)+5 = 4-4+5 = 5$$
$$u_3 = (3)^2-3(2)+5 = 9-6+5 = 8$$

$$u_3 - u_2 = 8-5 = 3 \qquad u_2 - u_1 = 5-4 = 1$$
$$u_3 - u_2 \neq u_2 - u_1$$

Thus, u_n is not an arithmetic sequence.

Example

The sum of the first n terms of an arithmetic series is given by $S_n = 4n - n^2$.
Find: **(i)** u_n **(ii)** u_{18}.
(iii) If $S_n = -60$, find the value of n.

Solution:

(i)
$$S_n = 4n - n^2$$
$$S_{n-1} = 4(n-1)-(n-1)^2$$
$$= 4(n-1)-(n^2-2n+1)$$
$$= 4n-4-n^2+2n-1$$
$$= -n^2+6n-5$$

$$u_n = S_n - S_{n-1}$$
$$= (4n-n^2)-(-n^2+6n-5)$$
$$= 4n-n^2+n^2-6n+5$$
$$= 5-2n$$

(ii)
$$u_n = 5-2n$$
$$u_{18} = 5-2(18)$$
$$= 5-36$$
$$= -31$$

(iii) Given: $S_n = -60$
$$\therefore \quad 4n-n^2 = -60$$
$$n^2-4n-60 = 0$$
$$(n+6)(n-10) = 0$$
$$n = -6 \quad \text{or} \quad n = 10$$
$$\therefore \quad n = 10, \text{ as } n \in \mathbb{N}.$$

If we need to find three consecutive terms of an arithmetic sequence, we let the numbers be:

$$a-d, \quad a, \quad a+d.$$

If we need to find five consecutive terms of an arithmetic sequence, we let the numbers be:

$$a-2d, \quad a-d, \quad a, \quad a+d, \quad a+2d.$$

Keep 'a' in the middle of the sequence.

Example

Three numbers are in arithmetic sequence. Their sum is 24 and their product is 494. Find the three numbers.

Solution:

Let the three terms be $(a-d)$, a, $(a+d)$, which are in arithmetic sequence.

Given: Sum of the three terms = 24

$$\therefore \quad (a-d)+a+(a+d)=24$$
$$a-d+a+a+d=24$$
$$3a=24$$
$$a=8$$

Given: Product of the three terms = 494

$$\therefore \quad (a-d)(a)(a+d)=494$$
$$(8-d)(8)(8+d)=494 \quad \text{(put in } a=8\text{)}$$
$$8(8-d)(8+d)=494$$
$$8(64-d^2)=494$$
$$512-8d^2=494$$
$$-8d^2=-18$$
$$8d^2=18$$
$$4d^2=9$$
$$d^2=\frac{9}{4}$$
$$d=\pm\sqrt{\frac{9}{4}}=\pm\frac{\sqrt{9}}{\sqrt{4}}=\pm\frac{3}{2}$$

$$a=8$$
$$a-d=8-\frac{3}{2}=\frac{13}{2}$$
$$a+d=8+\frac{3}{2}=\frac{19}{2}$$

Thus, the three terms are $\frac{13}{2}$, 8, $\frac{19}{2}$.

Exercise 7.3

1. The nth term of an arithmetic sequence is given by $u_n=5n-3$. Write down the first three terms.

2. The first three terms of an arithmetic series are $3+7+11+\cdots$
   Find: **(i)** u_{20} **(ii)** S_{10}.

15. How many terms are there in the arithmetic sequence 2, 5, 8, . . . 59?

16. Find the sum of the series: $-5 - 1 + 3 + \cdots + 151$.

17. Each of the following represents the first three terms of an arithmetic sequence.
In each case find the values of x, $x \in \mathbf{R}$.

(i) $x + 2, 11, 4x$ **(ii)** $2x - 1, 2x + 1, 3x$ **(iii)** $x + 1, 2x - 1, 5x + 3$
(iv) $5x - 1, 1, x + 1$ **(v)** $5x + 2, x^2, 3x - 2$ **(vi)** $3 - 5x, x^2, 3x + 1$

18. (a) If $4x + 5$, x^2, $2x - 5$ and y are four consecutive terms in an arithmetic sequence, find the values of x and y, $x, y \in \mathbf{R}$, and write down the four terms.

(b) The nth term of a sequence is $u_n = n^2 + 3n - 2$. Verify that the sequence is **not** arithmetic.

19. The sum of the first n terms, S_n, of some series is given below.
In each case, find u_n and verify that the series is arithmetic:

(i) $S_n = n^2 + 2n$ **(ii)** $S_n = n^2 - n$ **(iii)** $S_n = 2n^2 + 5n$ **(iv)** $S_n = 3n - 2n^2$

20. The sum of the first n terms of an arithmetic series is given by $S_n = 2n^2 - 3n$.
Find: **(i)** S_{20} **(ii)** u_n **(iii)** u_{20}.
(iv) If $S_n = 77$, find the value of n.

21. The nth term of an arithmetic sequence is given by $u_n = pn + q$.

(i) If $u_2 = -1$ and $u_5 = 17$, find the value of p and the value of q.

(ii) If $S_n = an^2 + bn$, find the value of a and the value of b.

22. Evaluate: **(i)** $\sum_{r=1}^{50} (2r + 1)$ **(ii)** $\sum_{r=1}^{60} (3r - 2)$.

23. The sum of the first n terms of an arithmetic series is given by $S_n = \frac{n}{2}(n + 1)$.
Evaluate: $\sqrt{S_{n+1} + S_n} - u_n$.

24. An arithmetic series has a common difference d, where $d > 0$.
Three consecutive terms of the series, $a - d$, a and $a + d$, have a sum of 24 and a product of 120.
Calculate the value of d.

25. Three numbers are in arithmetic sequence. Their sum is 24 and their product is 312.
Find two possible sets of values for these terms.

26. Three numbers are in arithmetic sequence. The middle number is 6.
The sum of their squares is 120.5. Find the other two numbers.

27. Five numbers are in arithmetic sequence. Their sum is 50. The product of the least and the greatest term is 64. Find the five numbers.

28. In an arithmetic series, $u_1 = \cos x$, $u_2 = 2 \sin x$, and $u_3 = 2 \cos x$, where $0 < x < \frac{\pi}{2}$.
Find x and hence the common difference, d, and the first term, a.

29. The nth term of a series is given by $u_n = \ln[2^{n-1}(x)]$.

(i) Write down the first three terms.

(ii) Verify that the series is arithmetic.

Geometric Sequences and Series

Consider the sequence of numbers 4, 12, 36, 108, ...
Each term, after the first, can be found by multiplying the previous term by 3.
This is an example of a geometric sequence.

> A sequence in which each term, after the first, is found by multiplying the previous term by a constant number is called a **geometric sequence**.

The first term in a geometric sequence is denoted by a.
The constant number, by which each term is multiplied, is called the **common ratio** and is denoted by r.

Note: $r \neq -1, 0, 1$

Consider the geometric series 3, 6, 12, 24, . . .

$$a = 3 \quad \text{and} \quad r = 2$$

Each term, after the first, is found by multiplying the previous term by 2.

Consider the geometric series 27, 9, 3, 1, . . .

$$a = 27 \quad \text{and} \quad r = \tfrac{1}{3}$$

Each term, after the first, is found by multiplying the previous term by $\frac{1}{3}$.

Note: Multiplying by $\frac{1}{3}$ is the same as dividing by 3.

In a geometric sequence, the common ratio, r, between any two consecutive terms is always the same.

$$\frac{\text{Any term}}{\text{Previous term}} = \frac{u_n}{u_{n-1}} = \text{constant} = r.$$

> If three terms, u_n, u_{n+1}, u_{n+2} are in geometric sequence, then:
> $$\frac{u_{n+2}}{u_{n+1}} = \frac{u_{n+1}}{u_n}.$$

General Term of a Geometric Sequence

In a geometric sequence, a is the first term and r is the common ratio.
Thus, in a geometric sequence:

$u_1 = a \qquad = a$

$u_2 = ar \qquad = ar$

$u_3 = (ar)r \quad = ar^2$

$u_4 = (ar^2)r = ar^3$ and so on.

Notice that the power of r is always **one less** than the term number.
Thus, the general term of a geometric sequence is given by:

$$\boxed{u_n = ar^{n-1}}$$

For example, $u_6 = ar^5$, $\quad u_{10} = ar^9$.

Note: If $u_n = pq^n$, where p and q are constants, the sequence is geometric.

Geometric Series

If the sequence $u_1, u_2, u_3, \ldots, u_n$ is geometric, then the corresponding series
$S_n = u_1 + u_2 + u_3 + \cdots + u_n$ is a geometric series.
The formula for S_n of a geometric series can be written in terms of the first term, a, and the common ratio, r.

> If $S_n = u_1 + u_2 + u_3 + \cdots + u_n$ is a geometric series, then:
>
> $$S_n = \frac{a(1-r^n)}{1-r} \text{ when } |r| < 1 \quad \text{or} \quad S_n = \frac{a(r^n - 1)}{r-1} \text{ when } |r| > 1.$$

Note: In practice it does not matter which form is used.

To derive this result:

$$\begin{aligned}
S_n &= a + ar + ar^2 + \cdots + ar^{n-1} \\
rS_n &= \phantom{a + {}} ar + ar^2 + \cdots + ar^{n-1} + ar^n \\
\hline
S_n - rS_n &= a \qquad\qquad\qquad\qquad\qquad - ar^n \qquad \text{(subtract)} \\
(1-r)S_n &= a - ar^n \\
(1-r)S_n &= a(1-r^n) \\
S_n &= \frac{a(1-r^n)}{1-r} \qquad\qquad (r \neq \pm 1)
\end{aligned}$$

Once we find the first term, a, and the common ratio, r, we can answer any question about a geometric sequence or series.

If we need to find three unknown consecutive terms in geometric sequence, we let the terms be:

$$\frac{a}{r}, \ a, \ ar.$$

Example

The nth term of a geometric sequence is $u_n = \left(\frac{2}{3}\right)^n$.

(i) Find the first three terms. (ii) Find S_5, the sum of the first five terms.

Solution:

(i)

$$u_n = \left(\frac{2}{3}\right)^n$$

$$u_1 = \left(\frac{2}{3}\right)^1 = \frac{2}{3}$$

$$u_2 = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$$

$$u_3 = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$$

Thus, the first three terms are:

$$\frac{2}{3}, \frac{4}{9}, \frac{8}{27}.$$

(ii)

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_5 = \frac{\frac{2}{3}\left[1-\left(\frac{2}{3}\right)^5\right]}{1-\frac{2}{3}}$$

$$= \frac{\frac{2}{3}\left[1-\frac{32}{243}\right]}{\frac{1}{3}}$$

$$= \frac{\frac{2}{3}\left(\frac{211}{243}\right)}{\frac{1}{3}} = \frac{422}{243}$$

Example

2, 6, 18, . . . , 1458 is a geometric sequence.

Find: (i) the nth term (ii) the number of terms in the sequence.

Solution:

(i) $a = 2$ (given)

$$r = \frac{u_2}{u_1} = \frac{6}{2} = 3$$

$$u_n = ar^{n-1}$$

$$u_n = 2(3)^{n-1}$$

We know that $a = 2$ and $r = 3$.
We need to find n, the number of terms.

(ii) **Given:** $u_n = 1458$

$$\therefore \quad 2(3)^{n-1} = 1458$$

$$3^{n-1} = 729$$

$$3^{n-1} = 3^6$$

$$n - 1 = 6$$

$$n = 7$$

Thus, there are 7 terms in the sequence.

Example

Three terms in geometric sequence are $x-3$, x, $3x+4$, where $x \in \mathbf{R}$.
Find two possible values of x.

Solution:

We use the fact that in a geometric sequence, any term divided by the previous term is always a constant.

Thus,
$$\frac{u_{n+2}}{u_{n+1}} = \frac{u_{n+1}}{u_n} \qquad \text{[common ratio]}$$
$$\frac{3x+4}{x} = \frac{x}{x-3} \qquad \text{[put in given values]}$$
$$(3x+4)(x-3) = (x)(x) \qquad \text{[multiply both sides by } (x)(x-3)\text{]}$$
$$3x^2 - 5x - 12 = x^2$$
$$2x^2 - 5x - 12 = 0$$
$$(2x+3)(x-4) = 0$$
$$2x+3 = 0 \quad \text{or} \quad x-4 = 0$$
$$x = -\frac{3}{2} \quad \text{or} \quad x = 4$$

To verify that a sequence is geometric, we must show the following:

$$\boxed{\frac{u_n}{u_{n-1}} = \text{constant.}}$$

Note: To show that a sequence is **not geometric**, it is necessary only to show that the ratio of any two consecutive terms is not the same. In practice, this usually involves showing that $u_3 \div u_2 \neq u_2 \div u_1$ or similar.

Example

Write down the first four terms of the sequence $u_n = 8\left(\frac{3}{4}\right)^n$ and show that the sequence is geometric.

Solution:

$$u_n = 8\left(\frac{3}{4}\right)^n$$

$$u_1 = 8\left(\frac{3}{4}\right)^1 = 8\left(\frac{3}{4}\right) = 6$$

$$u_2 = 8\left(\frac{3}{4}\right)^2 = 8\left(\frac{9}{16}\right) = \frac{9}{2}$$

$u_3 = 8\left(\frac{3}{4}\right)^3 = 8\left(\frac{27}{64}\right) = \frac{27}{8}$

$u_4 = 8\left(\frac{3}{4}\right)^4 = 8\left(\frac{81}{256}\right) = \frac{81}{32}$

Thus, the first four terms are 6, $\frac{9}{2}$, $\frac{27}{8}$, $\frac{81}{32}$.

$u_n = 8\left(\frac{3}{4}\right)^n \qquad u_{n-1} = 8\left(\frac{3}{4}\right)^{n-1}$

$$\frac{u_n}{u_{n-1}} = \frac{8(\frac{3}{4})^n}{8(\frac{3}{4})^{n-1}} = \frac{(\frac{3}{4})^n}{(\frac{3}{4})^{n-1}} = \left(\frac{3}{4}\right)^{n-(n-1)} = \left(\frac{3}{4}\right)^{n-n+1} = \left(\frac{3}{4}\right)^1 = \frac{3}{4} \qquad \text{(a constant)}$$

$\frac{u_n}{u_{n-1}}$ = a constant.

Thus, u_n is a geometric sequence.

Example

(i) In a geometric sequence, the second term is 8 and the fifth term is 64.
Find the first term, a, and the common ratio, r.

(ii) In a geometric sequence, the sum of the first and third terms is $\frac{20}{3}$ and the sum of the second and fourth terms is $\frac{20}{9}$.
Find the first term, a, and the common ratio, r.

Solution:

(i)

$$u_n = ar^{n-1}$$

Given: $u_2 = 8$
$\therefore$ $ar = 8$ ①

Given: $u_5 = 64$
$\therefore$ $ar^4 = 64$ ②

We now divide ② by ① to eliminate a and find r.

② ÷ ① gives:

$$\frac{ar^4}{ar} = \frac{64}{8}$$
$$r^3 = 8$$
$$r = 2$$

Put $r = 2$ into ① or ② to find a:

$$ar = 8 \quad ①$$
$$a(2) = 8$$
$$2a = 8$$
$$a = 4$$

Thus, the first term is $a = 4$ and the common ratio is $r = 2$.

Note: If the index of r is even, we get two values for r, one positive and the other negative.

(ii) $S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$

Given: $u_1 + u_3 = \frac{20}{3}$

$\therefore \quad a + ar^2 = \frac{20}{3}$

$a(1 + r^2) = \frac{20}{3}$ ①

Given: $u_2 + u_4 = \frac{20}{9}$

$\therefore \quad ar + ar^3 = \frac{20}{9}$

$ar(1 + r^2) = \frac{20}{9}$ ②

We now divide ② by ① to eliminate a and find r.

② ÷ ① gives:

$$\frac{ar(1+r^2)}{a(1+r^2)} = \frac{\frac{20}{9}}{\frac{20}{3}}$$

$$r = \frac{1}{3}$$

Put $r = \frac{1}{3}$ into ① or ② to find a.

$$a(1 + r^2) = \frac{20}{3}$$

$$a\left(1 + \frac{1}{9}\right) = \frac{20}{3}$$

$$a\left(\frac{10}{9}\right) = \frac{20}{3}$$

$$\frac{10}{9}a = \frac{20}{3}$$

$$10a = 60$$

(multiply both sides by 9)

$$a = 6$$

Thus, the first term is $a = 6$ and the common ratio is $r = \frac{1}{3}$.

Example

In an arithmetic sequence, the sum of the first term and the third term is 15. The first, third and seventh terms of the arithmetic sequence are the first three terms of a geometric sequence.

(i) Find the first term and the common difference of the arithmetic sequence, where the common difference is positive.

(ii) Find the first three terms and the common ratio of the geometric sequence.

Solution:

(i) For the arithmetic sequence, $u_n = a + (n-1)d$

$u_1 = a \qquad u_3 = a + 2d \qquad u_7 = a + 6d$

Given: $u_1 + u_3 = 15$

$\therefore \quad (a) + (a + 2d) = 15$

$a + a + 2d = 15$

$2a + 2d = 15$ ①

Given: u_1, u_3 and u_7 are the first three terms in a geometric sequence.

$$\therefore \quad \frac{u_7}{u_3} = \frac{u_3}{u_1} \qquad \text{[common ratio]}$$

$$\frac{a+6d}{a+2d} = \frac{a+2d}{a}$$

$$a(a+6d) = (a+2d)(a+2d) \qquad \text{[multiply both sides by } a(a+2d)\text{]}$$

$$a^2 + 6ad = a^2 + 4ad + 4d^2$$

$$6ad = 4ad + 4d^2$$

$$2ad - 4d^2 = 0$$

$$ad - 2d^2 = 0$$

$$d(a-2d) = 0$$

$$d = 0 \quad \text{or} \quad a - 2d = 0$$

$$d = 0 \quad \text{or} \quad a = 2d$$

Thus, $a = 2d$ ② (we are given $d > 0$)

We now solve between the simultaneous equations ① and ②.

$$2a + 2d = 15 \quad ①$$

$$2a + a = 15 \quad (a = 2d)$$

$$3a = 15$$

$$a = 5$$

$$2d = a \quad ②$$

$$2d = 5$$

$$d = \frac{5}{2}$$

(ii) For the geometric sequence:

$$u_1 = a = 5$$

$$u_2 = a + 2d = 5 + 2\left(\frac{5}{2}\right) = 5 + 5 = 10$$

$$u_3 = a + 6d = 5 + 6\left(\frac{5}{2}\right) = 5 + 15 = 20$$

$$r = \frac{u_2}{u_1} = \frac{10}{5} = 2$$

Thus, the first three terms of the geometric sequence are 5, 10 and 20 and the common ratio is 2.

Exercise 7.4

1. The first three terms of a geometric series are $2 + 6 + 18 + \cdots$.

(i) Express, in terms of n: **(a)** u_n **(b)** S_n.

(ii) Find: **(a)** u_8 and **(b)** S_8.

2. The first three terms of a geometric series are $64 - 32 + 16$.

(i) Find, in terms of n: **(a)** u_n **(b)** S_n.

(ii) Find: **(a)** u_{10} and **(b)** S_{10}.

For each of the following geometric sequences, find u_n, the nth term:

3. 5, 10, 20, . . . **4.** 4, 12, 36, . . . **5.** 27, 18, 12, . . .

6. 50, −20, 8, . . . **7.** $1, 2a, 4a^2, \ldots$ **8.** $\frac{5}{a}, \frac{10}{a^2}, \frac{20}{a^3}, \ldots$

9. Verify that the sequence $u_n = 5^n$ is geometric.

10. Verify that the sequence $u_n = 2(3)^{n+1}$ is geometric.

11. Verify that the sequence $u_n = n^2 - 3$ is not geometric.

12. The sum to n terms of a series is given by $3(2^n - 1)$.
(i) Find u_n, the nth term. **(ii)** Verify that the series is geometric.

Find, in terms of n, the sum of the first n terms of the geometric series:

13. $6 + 12 + 24 + \cdots$ **14.** $6 + 4 + \frac{8}{3} + \cdots$ **15.** $63 - 21 + 7 - \cdots$

16. Find, in terms of n, the sum of the first n terms of the geometric series $18 + 12 + 8 + \cdots$.
If $S_n = \frac{1330}{27}$, find the value of n.

17. If $\sum_{r=1}^{n} 2^{n+1} = 508$, find the value of n.

18. A geometric series has 6 terms, a common ratio of $\frac{1}{2}$ and a sum of $\frac{189}{8}$.

Find: **(i)** the first term **(ii)** the nth term.

19. The lengths of the sides of a triangle are in geometric sequence. The length of the shortest side is 4 cm and the perimeter of the triangle is 19 cm. Find the lengths of the other sides.

Each of the following represents the first three terms of a geometric sequence.
In each case find the value(s) of x, $x \in \mathbf{R}$:

20. $x - 2, x, x + 3$ **21.** $x - 1, 2x + 1, 4x + 17$

22. $4x + 36, 2x + 6, x$ **23.** $x - 6, 2x, 8x + 20$

24. $x + 1, x + 4, 3x + 2$ **25.** $3x - 5, x - 1, x - 2$

26. $x + 1, x - 1$ and $2x - 5$ are the first three terms of a geometric series.
(i) Find two values for x.
(ii) Write down the first four terms of the two resulting series.

27. Four terms in geometric sequence are: $6, a, b, \frac{3}{4}$. Find the values of a and b.

28. The third term, u_3, of a geometric sequence is −63. The fourth term, u_4, is 189.
Find: **(i)** the common ratio **(ii)** the first term.
Express, in terms of n: **(iii)** u_n **(iv)** S_n.

29. In a geometric series, the fourth term is 12 and the seventh term is 324.
Find: **(i)** the nth term **(ii)** S_7, the sum of the first seven terms.

30. $S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$ is a geometric series.
$u_3 - u_2 = 5$ and $u_4 - u_3 = 6$.
Find the common ratio, r, and the first term, a.

31. A geometric series has a common ratio r.
The first three terms of the series are $\frac{a}{r}$, a and ar.
The product of the three terms is 216 and the sum of the three terms is 21.
Find: **(i)** the value of a **(ii)** the values of r.
(iii) Write down the first three terms.

32. The product of the first three terms of a geometric series is 27 and the sum of these terms is 13. Find the first four terms of the series.

33. Three terms, a, b and $a + b$, are in arithmetic sequence.
Three terms, a, b and ab, are in geometric sequence.
Find the value of a and the value of b, where $a, b \in \mathbf{R}$ and $a, b \neq 0$.

34. p, 10 and q are consecutive terms of an arithmetic sequence.
1, p and q are consecutive terms of a geometric sequence.
Find the value of p and the value of q, $p, q \in \mathbf{R}$.

35. The first, fifth and seventeenth terms of an arithmetic series are the first three terms of a geometric series. The sum of the first four terms of the arithmetic series is 28. Find the common difference of the arithmetic series and the common ratio of the geometric series.

36. The first, fifth and twenty-first terms of an arithmetic sequence are the first three terms of a geometric sequence. Find the common ratio of the geometric sequence.

37. p, m and q are three consecutive terms of an arithmetic sequence.
p, n and q are three consecutive terms of a geometric sequence, where $p, q, n > 0$.
Show that $m \geqslant n$.

Infinite Geometric Series

When a series has an infinite number of terms, it is called an **infinite series** and the sum of the series is called the **sum to infinity** of the series.

Let us consider the value of a proper fraction (less than 1) if we keep multiplying it by itself. Take for example, $\frac{1}{4}$, and keep multiplying it by itself, i.e. $(\frac{1}{4})^n$, as n increases indefinitely. We can represent this situation in a table using a calculator.

n	1	2	3	...	10
$(\frac{1}{4})^n$	0.25	0.0625	0.015625	...	0.0000009537

From the table we can see that the bigger the value of n, the nearer $(\frac{1}{4})^n$ gets to 0.
(This will happen for any proper fraction, positive or negative.)
We say that the limit of $(\frac{1}{4})^n$, as n approaches infinity, is 0.

Symbolically: $\boxed{\lim_{n\to\infty} (\text{proper fraction})^n = 0}$

$n\to\infty$ means 'as n approaches infinity'.
lim is short for limit.
In general, for the infinite geometric series:

$$a + ar + ar^2 + ar^3 + \cdots$$

if r is a proper fraction, then the terms will get closer to zero.
For r to be a proper fraction it must be between -1 and 1, i.e., $-1 < r < 1$.

$$\therefore \quad \text{If } -1 < r < 1$$
$$\text{then} \quad \lim_{n\to\infty} r^n = 0$$

Notes: If $r > 1$ or $r < -1$, then $\lim_{n\to\infty} r^n$ does not exist.

The sum to infinity, S_∞, of a series is denoted by $\lim_{n\to\infty} S_n$.

If $\lim_{n\to\infty} S_n$ exists, the series is said to be **convergent**.

If $\lim_{n\to\infty} S_n$ does **not** exist, the series is said to be **divergent**.

Let us now develop the general formula for the sum to infinity of a geometric series in which $-1 < r < 1$.

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

The only part of this formula that changes as n increases is r^n.
As, $n\to\infty$, $r^n\to 0$, because r is a proper fraction.

$$\therefore \quad S_\infty = \frac{a(1-0)}{1-r} = \frac{a}{1-r}$$

Sum to infinity of a geometric series
$S_\infty = \dfrac{a}{1-r} = \dfrac{\text{first term}}{1 - \text{common ratio}}$
if $-1 < r < 1$.

Note: $-1 < r < 1$ is often written $|r| < 1$.

Example

(i) Find the sum to infinity of the geometric series: $1+\left(\frac{2}{5}\right)+\left(\frac{2}{5}\right)^2+\left(\frac{2}{5}\right)^3+\cdots$.

(ii) Evaluate $\sum_{n=0}^{\infty}\left(\frac{5}{2x+1}\right)^n$, in terms of x, where $x>2$.

Solution:

(i) $1+\left(\frac{2}{5}\right)+\left(\frac{2}{5}\right)^2+\left(\frac{2}{5}\right)^3+\cdots$

This is an infinite geometric series with first term $a=1$ and common ratio $r=\frac{2}{5}$.

$$S_\infty=\frac{a}{1-r}=\frac{1}{1-\frac{2}{5}}=\frac{5}{5-2}=\frac{5}{3}$$

(ii) $\sum_{n=0}^{\infty}\left(\frac{5}{2x+1}\right)^n=1+\left(\frac{5}{2x+1}\right)+\left(\frac{5}{2x+1}\right)^2+\left(\frac{5}{2x+1}\right)^3+\cdots$

This is an infinite geometric series with first term $a=1$ and common ratio $r=\frac{5}{2x+1}$.

$$S_\infty=\frac{a}{1-r}=\frac{1}{1-\frac{5}{2x+1}}=\frac{2x+1}{2x+1-5}=\frac{2x+1}{2x-4}$$

Example

$\sum_{n=0}^{\infty}(2x-3)^n=1+(2x-3)+(2x-3)^2+(2x-3)^3+\cdots$ is a geometric series.

(i) Find, in terms of x, the sum to infinity.

(ii) If the sum to infinity is $\frac{5}{4}$, find the value of x.

(ii) Find the range of values of x for which the sum to infinity exists.

Solution:

This is an infinite geometric series with first term $a=1$ and common ratio $r=(2x-3)$.

(i)
$$S_\infty=\frac{a}{1-r}=\frac{1}{1-(2x-3)}=\frac{1}{1-2x+3}=\frac{1}{4-2x}$$

(ii) Given: $S_\infty=\frac{5}{4}$

$$\therefore\ \frac{1}{4-2x}=\frac{5}{4}$$
$$4=20-10x$$
$$10x=16$$
$$5x=8$$
$$x=\frac{8}{5}$$

(iii) S_∞ exists if $-1 < r < 1$ (i.e. $|r| < 1$)

$$-1 < r < 1$$

$$-1 < 2x - 3 < 1 \quad (r = 2x - 3)$$

$$2 < 2x < 4 \quad \text{(add 3 to each part)}$$

$$1 < x < 2 \quad \text{(divide each part by 2)}$$

Thus, the sum to infinity exists for $1 < x < 2$.

Example

The sum to infinity of a geometric series is 36 and the second term of the series is 8. Find two possible series.

Solution:

Let the series be $a + ar + ar^2 + \cdots$

Given: $S_\infty = 36$

$$\therefore \quad \frac{a}{1-r} = 36$$

$$a = 36(1 - r) \quad ①$$

Given: $u_2 = 8$

$$\therefore \quad ar = 8 \quad ②$$

We now solve between ① and ②:

$$ar = 8 \quad ②$$

$$36(1 - r)r = 8$$

[replace a with $36(1 - r)$]

$$(36 - 36r)r = 8$$

$$36r - 36r^2 = 8$$

$$-36r^2 + 36r - 8 = 0$$

$$36r^2 - 36r + 8 = 0$$

$$9r^2 - 9r + 2 = 0$$

$$(3r - 2)(3r - 1) = 0$$

$$3r - 2 = 0 \quad \text{or} \quad 3r - 1 = 0$$

$$3r = 2 \quad \text{or} \quad 3r = 1$$

$$r = \tfrac{2}{3} \quad \text{or} \quad r = \tfrac{1}{3}$$

Put $r = \frac{2}{3}$ and $r = \frac{1}{3}$ into ① or ② to find the value of a.

$$a = 36(1 - r) \quad ①$$

$r = \frac{2}{3}$	$r = \frac{1}{3}$
$a = 36\left(1 - \frac{2}{3}\right)$	$a = 36\left(1 - \frac{1}{3}\right)$
$a = 36\left(\frac{1}{3}\right)$	$a = 36\left(\frac{2}{3}\right)$
$a = 12$	$a = 24$

Thus, we have two series which obey the two given conditions:

(i) $a = 12$, $r = \frac{2}{3}$, the series is $12 + 8 + 5\frac{1}{3} + \cdots$

(ii) $a = 24$, $r = \frac{1}{3}$, the series is $24 + 8 + 2\frac{2}{3} + \cdots$.

Recurring Decimals

An application of the sum of infinite geometric series is expressing non-terminating, recurring decimals as rational numbers.

Note: The first five letters in the word 'rational' spell 'ratio'. In other words, a rational number is any number that can be written as a ratio (i.e. a fraction).

Recurring decimals can be expressed neatly by placing a dot over the first and last figures which repeat.
This is called the **dot notation**. For example:

1. $0.\dot{4} = 0.44444\ldots = \frac{4}{9}$

2. $0.1\dot{6} = 0.166666\ldots = \frac{1}{6}$

3. $1.\dot{2}\dot{5} = 1.252525\ldots = 1 + \frac{25}{99} = \frac{124}{99}$

4. $0.\dot{1}8\dot{5} = 0.185185185\ldots = \frac{5}{27}$

Example

Express the recurring decimal $0.7\dot{3}$ in the form $\frac{a}{b}$, where $a, b \in \mathbb{N}$.

Solution:

$$0.7\dot{3} = 0.733333\ldots$$

$$= 0.7 + 0.03 + 0.003 + 0.0003 + \cdots$$

$$= \frac{7}{10} + \left[\frac{3}{100} + \frac{3}{1{,}000} + \frac{3}{10{,}000} + \cdots\right]$$

The series in the brackets is an infinite geometric series, with $a = \frac{3}{100}$ and $r = \frac{1}{10}$.

$$S_\infty = \frac{a}{1-r} = \frac{\frac{3}{100}}{1-\frac{1}{10}} = \frac{3}{100-10} = \frac{3}{90} = \frac{1}{30}$$

Thus, $0.7\dot{3} = \frac{7}{10} + \frac{1}{30} = \frac{22}{30} = \frac{11}{15}$.

Exercise 7.5

Find the sum to infinity of each of the following geometric series:

1. $1 + \frac{1}{2} + \frac{1}{4} + \cdots$

2. $2 + \frac{2}{3} + \frac{2}{9} + \cdots$

3. $5 + 1 + \frac{1}{5} + \cdots$

4. $2 + 1.8 + 1.62 + \cdots$

5. $3 - \frac{3}{2} + \frac{3}{4} - \cdots$

6. $\frac{3}{2} - \frac{1}{4} + \frac{1}{24} - \cdots$

7. $0.2 - 0.1 + 0.05 - \cdots$

8. $1 + x + x^2 + \cdots$

9. $2 + \frac{2a}{3} + \frac{2a^2}{9} + \cdots$

10. Evaluate: **(i)** $\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n$ **(ii)** $\sum_{n=0}^{\infty}3\left(\frac{1}{2}\right)^n$ **(iii)** $\sum_{n=0}^{\infty}x(1-x)^n$.

11. Find the sum to infinity of the geometric series $\frac{5}{10}+\frac{5}{100}+\frac{5}{1,000}+\cdots$.
Using this series, show that $1.\dot{5}=\frac{14}{9}$.

Express each of the following recurring decimals in the form $\frac{a}{b}$, where $a, b \in \mathbf{N}$:

12. $0.\dot{4}$ **13.** $2.\dot{2}$ **14.** $0.\dot{7}\dot{2}$ **15.** $0.2\dot{7}$ **16.** $0.1\dot{2}$ **17.** $1.8\dot{3}$

18. The sum to infinity of a geometric series is 20 and the common ratio is $\frac{2}{5}$. Find the first term.

19. The sum to infinity of a geometric series is 4 and the first term is 6. Find the common ratio.

20. The sum to infinity of a geometric series is 5. The common ratio and the first term of the series are equal. Find the common ratio.

21. If the sum to infinity of a geometric series is five times the first term, find the common ratio.

22. $24x-5$, $6x-1$ and x are the first three terms of a geometric series.
Find two values for: **(i)** x **(ii)** the common ratio **(iii)** the sum to infinity.

23. In a geometric series, $u_1=a$ and $u_2=a^2-2a$, $a\neq 0$.
(i) Write down, in terms of a, the common ratio of the series.
(ii) If the sum to infinity is 5, find the value of a.
(iii) Find the range of values of $a \in \mathbf{R}$ for which the series has a sum to infinity.

24. $\sum_{n=0}^{\infty}x^2\left(\frac{1}{1-x}\right)^n=x^2+\frac{x^2}{1-x}+\frac{x^2}{(1-x)^2}+\frac{x^2}{(1-x)^3}+\cdots$ is an infinite geometric series.
(i) Find, in terms of x: **(a)** the common ratio **(b)** the sum to infinity.
(ii) If the sum to infinity is 30, find the values of x.

25. **(i)** Factorise $1-r^2$.
(ii) $a+ar+ar^2+ar^3+\cdots$ is an infinite geometric series with $|r|<1$.
The sum to infinity is 8 and the sum to infinity of the even terms is 2.
Find the value of the common ratio.

26. The sum to infinity of a geometric series is 27 and the second term of the series is 6.
Find two possible series.

27. The sum to infinity of a geometric series is 2. When the terms of this geometric sequence are squared a new geometric sequence is obtained whose sum to infinity is 12.
Find the first term, a, and the common ratio, r, where $|r|<1$.

Series of the Form $\sum\limits_{n=0}^{\infty} nx^n$

A series can be formed by multiplying, term by term, the terms of an arithmetic series and a geometric series, called '**Arithmetic–geometric**' series.

Consider the series: $1 + 3x + 5x^2 + 7x^3 + \cdots$

Each term consists of two parts:

1. A coefficient: $1, 3, 5, 7, \ldots,$ which are in arithmetic sequence.
2. A power: $1, x, x^2, x^3, \ldots,$ which are in geometric sequence.

To find a formula for S_n for an arithmetic–geometric series, do the following:

1. **Write down S_n.**
2. **Write down rS_n (multiply by the common ratio for the powers).**
3. **Subtract.**
4. **Use the formula for the sum of the first n terms of a geometric series, $\dfrac{a(1-r^n)}{1-r}$.**
5. **Divide both sides by $(1-r)$.**

Example

$S_n = \sum\limits_{r=1}^{n} (r+3)x^{r-1}$, where $|x| < 1$.

(i) Write S_n in terms of n. **(ii)** Find: $\sum\limits_{r=1}^{\infty} (r+3)x^{r-1}$.

(iii) Evaluate: $\sum\limits_{r=1}^{\infty} (r+3)\left(\frac{1}{3}\right)^{r-1}$.

Solution:

$$S_n = \sum_{r=1}^{n} (r+3)x^{r-1} = 4 + 5x + 6x^2 + 7x^3 + \cdots + (n+3)x^{n-1}$$

(i)

[Multiply S_n by x, the common ratio of the geometric series]

$$\begin{aligned} S_n &= 4 + 5x + 6x^2 + 7x^3 + \cdots + (n+3)x^{n-1} \\ xS_n &= \phantom{4 + {}} 4x + 5x^2 + 6x^3 + \cdots + (n+2)x^{n-1} + (n+3)x^n \\ \hline (1-x)S_n &= 4 + (x + x^2 + x^3 + \cdots + x^{n-1}) - (n+3)x^n \quad \text{(subtract)} \end{aligned}$$

[The series in the brackets is a geometric series with $a = x$, $r = x$ and $(n-1)$ terms.
Thus, S_n for this series $= \dfrac{a(1-r^{n-1})}{1-r} = \dfrac{x(1-x^{n-1})}{1-x}$.]

$$(1-x)S_n = 4 + (x + x^2 + x^3 + \cdots + x^{n-1}) - (n+3)x^n$$

$$\therefore \quad (1-x)S_n = 4 + \frac{x(1-x^{n-1})}{1-x} - (n+3)x^n$$

$$S_n = \frac{4}{(1-x)} + \frac{x(1-x^{n-1})}{(1-x)^2} - \frac{(n+3)x^n}{(1-x)}$$

(ii) If $|x| < 1$, then $\lim\limits_{n\to\infty} x^n = 0$ and $\lim\limits_{n\to\infty} x^{n-1} = 0$.

$$\sum_{r=1}^{\infty}(r+3)x^{r-1} = \lim_{n\to\infty} S_n$$

$$S_n = \frac{4}{(1-x)} + \frac{x(1-x^{n-1})}{(1-x)^2} - \frac{(n+3)x^n}{(1-x)}$$

$$\text{Thus, } \lim_{n\to\infty} S_n = \frac{4}{(1-x)} + \frac{x(1-0)}{(1-x)^2} - \frac{(n+3)(0)}{(1-x)}$$

$$= \frac{4}{(1-x)} + \frac{x}{(1-x)^2} = \frac{4(1-x)+x}{(1-x)^2} = \frac{4-3x}{(1-x)^2}$$

(iii) $\displaystyle\sum_{r=1}^{\infty}(r+3)\left(\tfrac{1}{3}\right)^{r-1} = 4 + 5\left(\tfrac{1}{3}\right) + 6\left(\tfrac{1}{3}\right)^2 + 7\left(\tfrac{1}{3}\right)^3 + \cdots + (r+3)\left(\tfrac{1}{3}\right)^{r-1} + \cdots$

This is exactly the same as the original series with $x = \frac{1}{3}$.

$$\text{But } \sum_{r=1}^{\infty}(r+3)x^{r-1} = \frac{4-3x}{(1-x)^2}$$

$$\therefore \quad \sum_{r=1}^{\infty}(r+3)\left(\tfrac{1}{3}\right)^{r-1} = \frac{4-3\left(\frac{1}{3}\right)}{\left(1-\frac{1}{3}\right)^2} \qquad \left[\text{put in } x = \tfrac{1}{3} \text{ into } \frac{4-3x}{(1-x)^2}\right]$$

$$= \frac{4-1}{\frac{4}{9}} = \frac{36-9}{4} = \frac{27}{4}$$

Sometimes u_n is not given.

Example

For the series $1 \,.\, 1 + 3 \,.\, 2 + 5 \,.\, 2^2 + 7 \,.\, 2^3 + \cdots$, find, in terms of n:

(i) u_n, the nth term **(ii)** S_n, the sum of the first n terms.

Hence, evaluate S_{10}.

Solution:

(i) $1 \,.\, 1 + 3 \,.\, 2 + 5 \,.\, 2^2 + 7 \,.\, 2^3 + \cdots$

This series is a combination of an arithmetic series and a geometric series.
Thus, we need to find u_n separately for each of these series and combine the results.

Arithmetic series: $1+3+5+7+\cdots$

$u_n=a+(n-1)d$
$=1+(n-1)2$
$=2n-1$

Geometric series: $1+2+2^2+2^3+\cdots$

$u_n=ar^{n-1}$
$=1\,.\,2^{n-1}$
$=2^{n-1}$

Thus, the nth term of the series is given by $u_n=(2n-1)2^{n-1}$.

(ii) [Multiply S_n by 2, the common ratio of the geometric series]

$$S_n=1\,.\,1+3\,.\,2+5\,.\,2^2+7\,.\,2^3+\cdots+(2n-1)2^{n-1}$$
$$2S_n=\quad 1\,.\,2+3\,.\,2^2+5\,.\,2^3+\cdots+(2n-3)2^{n-1}+(2n-1)2^n$$
$$-S_n=1+2\,.\,2+2\,.\,2^2+2\,.\,2^3+\cdots+2\,.\,2^{n-1}-(2n-1)2^n$$
$$-S_n=1+2(2+2^2+2^3+\cdots+2^{n-1})-(2n-1)2^n$$

[The series in the brackets is a geometric series with $a=2$ $r=2$, and $(n-1)$ terms.
Thus, S_n for this series $=\dfrac{a(r^{n-1}-1)}{r-1}=\dfrac{2(2^{n-1}-1)}{2-1}=2^n-2$.]

$$-S_n=1+2(2+2^2+2^3+\cdots+2^{n-1})-(2n-1)2^n$$
$$-S_n=1+2(2^n-2)-(2n-1)2^n$$
$$S_n=(2n-1)2^n-2(2^n-2)-1$$
$$=(2n-1)2^n-2\,.\,2^n+4-1$$
$$=(2n-1-2)2^n+3$$
$$=(2n-3)2^n+3$$

Exercise 7.6

1. (i) Find S_n, the sum to n terms of the series $1+x+x^2+x^3+\cdots+x^{n-1}$.

(ii) Complete the following table:

$S_n=1+2x+3x^2+4x^3+\cdots+nx^{n-1}$ $xS_n=$
$(1-x)S_n=$
$S_n=$
$\lim\limits_{n\to\infty} S_n=$
$1+2(\frac{1}{5})+3(\frac{1}{5})^2+4(\frac{1}{5})^3+\cdots+n(\frac{1}{5})^{n-1}+\cdots=\boxed{}$

(iii) Write down the first four terms of the series: $\sum_{r=1}^{n} rx^{2r-2}$.

Hence find, in terms of x, an expression for $\sum_{r=1}^{\infty} rx^{2r-2}$.

2. For the series $2x+3x^2+4x^3+\cdots+(n+1)x^n$, where $-1<x<1$:

(i) write S_n, in terms of n **(ii)** find $\sum_{r=1}^{\infty}(r+1)x^r$

(iii) evaluate $\sum_{r=1}^{\infty}(r+1)\left(\frac{1}{2}\right)^r$.

3. If $S_n=3+5x+7x^2+\cdots+(2n+1)x^{n-1}$:

(i) show that $S_n=\dfrac{3-(2n+1)x^n}{1-x}+\dfrac{2x(1-x^{n-1})}{(1-x)^2}$

(ii) if $|x|<1$, find $\sum_{n=1}^{\infty}(2n+1)x^{n-1}$ and evaluate $\sum_{n=1}^{\infty}(2n+1)\left(\frac{1}{3}\right)^{n-1}$.

4. Find an expression for S_n of the series:

$2+5x+8x^2+\cdots+(3n-1)x^{n-1}$.

If $|x|<1$, find $\lim_{n\to\infty} S_n$. If $\sum_{n=1}^{\infty}(3n-1)x^{n-1}=4$, find the value of x.

5. Let $f(x)=\sum_{n=1}^{\infty}(2n-1)x^{n-1}=1+3x+5x^2+7x^3+\cdots$ for $-1<x<1$.

Show that $f(x)=\dfrac{1+x}{(1-x)^2}$. Evaluate $\sum_{n=1}^{\infty}(2n-1)\left(\frac{1}{4}\right)^{n-1}$.

6. Let $f(x)=\sum_{n=1}^{\infty}(4n-3)x^{n-1}$, where $|x|<1$.

(i) Show that $f(x)=\dfrac{1+3x}{(1-x)^2}$. **(ii)** Evaluate $\sum_{n=1}^{\infty}(4n-3)\left(\frac{4}{5}\right)^{n-1}$.

7. For the series $2\,.\,2+3\,.\,2^2+4\,.\,2^3+\cdots+(n+1)2^n$, find:

(i) S_n, the sum of the first n terms **(ii)** S_9, the sum of the first nine terms.

8. Write out the first four terms of the series $\sum_{r=1}^{n} r2^{r-1}$.

Show that: $\sum_{r=1}^{n} r2^{r-1}=(n-1)2^n+1$. Evaluate $\sum_{r=1}^{8} r2^{r-1}$.

9. For the series $2\,.\,1+3\,.\,3+4\,.\,3^2+5\,.\,3^3+\cdots$, show that $S_n=\dfrac{(2n+1)3^n-1}{4}$.

10. Evaluate $\sum_{n=1}^{\infty} n\left(\frac{1}{3}\right)^n$.