**Exercise 2.7**

1. If $1 + 2i$ is a root of $z^3 + 2z^2 - 3z + 20 = 0$, find the other two roots.

2. If $-2 + 3i$ is a root of $z^3 - z^2 - 7z - 65 = 0$, find the other two roots.

3. Show that $z = 2$ is a root of the equation $z^3 - 8z^2 + 46z - 68 = 0$ and find the other two roots.

4. (i) Express in the form $a + bi$: (a) $(1 + i)^2$  (b) $(1 + i)^3$.

(ii) Show that $1 + i$ is a root of $z^3 - 5z^2 + 8z - 6 = 0$ and find the other two roots.

5. Verify that $1 + 3i$ is a root of the equation $z^3 - 7z^2 + 20z - 50 = 0$ and find the other roots.

6. Verify that $-2 + 2i$ is a root of the equation $z^3 + 3z^2 + 4z - 8 = 0$ and find the other roots.

7. Verify that $2 - i$ is a root of the equation $z^3 - 11z + 20 = 0$ and find the other roots.

8. Verify that $2 + 3i$ is a root of the equation $2z^3 - 9z^2 + 30z - 13 = 0$ and find the other roots.

9. Verify that $i$ is a root of the equation $z^3 - iz^2 - 9z + 9i = 0$ and find the other roots.

10. (i) Express in the form $a + bi$: (a) $(1 - i)^2$  (b) $(1 - i)^3$.

(ii) If $1 - i$ is a root of the equation $z^3 - 4z^2 + 6z + k = 0$, $k \in \mathbb{R}$, find the value of $k$ and the other roots.

11. If $1 - 2i$ is a root of the equation $z^3 + kz^2 + 7z + k - 2 = 0$, $k \in \mathbb{R}$, find the value of $k$ and the other roots.

12. One root of the equation $z^3 + az^2 + bz - 4 = 0$, $a, b \in \mathbb{R}$, is $1 + i$.

Find the value of $a$ and the value of $b$.

13. One root of the equation $z^3 + pz^2 + q = 0$, $p, q \in \mathbb{R}$, is $4 - i$.

Find the value of $p$ and the value of $q$.

14. If $(z - 2)(z^2 + az + b) = z^3 - 4z^2 + 6z - 4$, $a, b \in \mathbb{Z}$, find the value of $a$ and the value of $b$.

15. Let $p(z) = z^3 + (4 - 2i)z^2 + (5 - 8i)z - 10i$, where $i^2 = -1$.

Determine the real numbers $a$ and $b$ if $p(z) = (z - 2i)(z^2 + az + b)$.

16. (i) Factorise $z^2 - 5z + 6$ and, hence, solve the equation $z^2 - 5z + 6 = 0$.

(ii) Show that $z^2 - 5z + 6$ is a factor of $z^3 + (-4 + i)z^2 + (1 - 5i)z + 6(1 + i)$.

(iii) Find the three roots of the equation $z^3 + (-4 + i)z^2 + (1 - 5i)z + 6(1 + i) = 0$.

17. (i) Factorise $z^2 - 4$ and, hence or otherwise, solve the equation $z^2 - 4 = 0$.

(ii) Show that $z^2 - 4$ is a factor of $z^3 + (3 + i)z^2 - 4z - 4(3 + i)$.

(iii) Find the three roots of the equation $z^3 + (3 + i)z^2 - 4z - 4(3 + i) = 0$.

18. $ki$ is a root of the equation $2z^3 - z^2 + 18z - 9 = 0$, $k \in \mathbb{R}$.

Find the values of $k$ and the three roots of the equation $2z^3 - z^2 + 18z - 9 = 0$.
Polar Coordinates and the Polar Form of a Complex Number

Polar Coordinates

Consider the complex number $z = x + yi$. The position of $z$ on the Argand diagram can be given by Cartesian, or rectangular coordinates, $(x, y)$. An alternative way of describing the position of $z$ is to give its modulus, $r$ and its argument, $\theta$.

$(r, \theta)$ are called the polar coordinates of the complex number.

1. Modulus

The modulus $r = |z| = \sqrt{x^2 + y^2}$.

The modulus is the distance from the origin to the point representing the complex number on the Argand diagram.

2. Argument

The argument $= \theta$ (usually in radians).

The argument, $\theta$, is the angle between the positive real-axis and the line from the origin to the point $z = x + yi$.

Note: Drawing a diagram can be a very good aid in calculating $\theta$.

Polar Form

Having calculated $r$ and $\theta$, there is a simple connection between the Cartesian coordinates $(x, y)$ and the polar coordinates $(r, \theta)$.

$$\frac{x}{r} = \cos \theta \quad \frac{y}{r} = \sin \theta$$

$$x = r \cos \theta \quad y = r \sin \theta$$

Now we can write $z = x + yi$ in terms of $r$ and $\theta$.

$$z = x + yi$$

$$z = (r \cos \theta) + (r \sin \theta)i$$

$$z = r(\cos \theta + i \sin \theta)$$

This is called the polar form of the complex number.

The polar form of the complex number $z = x + yi$ is:

$$z = r(\cos \theta + i \sin \theta).$$

Note: It is conventional to write $i$ before $\sin \theta$.

In other words, $i \sin \theta$ is preferable to $\sin \theta i$. 
Example

Express each of these complex numbers in the form \( r(\cos \theta + i \sin \theta) \), where \( i^2 = -1 \):

(i) \(-1 + i\)
(ii) \(-\sqrt{3} - i\)
(iii) \(\frac{1}{2} + \frac{\sqrt{3}}{2}i\)
(iv) \(-6i\)

Solution:

(i) \(-1 + i = (-1, 1)\)
\[ r = |-1 + i| = \sqrt{(-1)^2 + (1)^2} = \sqrt{1 + 1} = \sqrt{2} \]
\[ \tan \alpha = \frac{1}{-1} = -1 \quad \Rightarrow \quad \alpha = \frac{\pi}{4} \]
\[ \therefore \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4} \]
\[ -1 + i = \sqrt{2} \left( \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right) \]

(ii) \(-\sqrt{3} - i = (-\sqrt{3}, -1)\)
\[ r = |-\sqrt{3} - i| = \sqrt{(-\sqrt{3})^2 + (-1)^2} = \sqrt{3 + 1} = \sqrt{4} = 2 \]
\[ \tan \alpha = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} = \frac{\pi}{6} \]
\[ \therefore \theta = \pi + \frac{\pi}{6} = \frac{7\pi}{6} \]
\[ -\sqrt{3} - i = 2 \left( \cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6} \right) \]

(iii) \(\frac{1}{2} + \frac{\sqrt{3}}{2}i = \left( \frac{1}{2} \right) \left( \frac{\sqrt{3}}{2} \right)\)
\[ r = \left| \frac{1}{2} + \frac{\sqrt{3}}{2}i \right| = \sqrt{\left( \frac{1}{2} \right)^2 + \left( \frac{\sqrt{3}}{2} \right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{1} = 1 \]
\[ \tan \theta = \frac{\sqrt{3}}{1} = \sqrt{3} \quad \Rightarrow \quad \theta = \frac{\pi}{3} \]
\[ \therefore \frac{1}{2} + \frac{\sqrt{3}}{2}i = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \]
(iv) \(-6i = 0 - 6i = (0, -6)\)
\[ r = |0 - 6i| = \sqrt{0^2 + (-6)^2} = \sqrt{0 + 36} = \sqrt{36} = 6 \]
\[ \theta = \frac{3\pi}{2} \]
\[ \therefore -6i = 6 \left( \cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} \right) \]

Sometimes we are given a number in polar form \(r(\cos \theta + i \sin \theta)\), and asked to write it in Cartesian form, \(x + yi\). Again, it is good practice to draw a diagram.

**Example**

Express \(4 \left( \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right)\) in the form \(x + yi\).

Solution:
\(\frac{5\pi}{6}\) is in the 2nd quadrant, so:
\[ \cos \frac{5\pi}{6} = -\cos \frac{\pi}{6} = -\frac{\sqrt{3}}{2} \]
\[ \sin \frac{5\pi}{6} = \sin \frac{\pi}{6} = \frac{1}{2} \]
\[ \therefore 4 \left( \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right) = 4 \left( \frac{\sqrt{3}}{2} + i \frac{1}{2} \right) = -2\sqrt{3} + 2i \]

**Exercise 2.8**

Express each of the following complex numbers in the form \(r(\cos \theta + i \sin \theta)\), where \(i^2 = -1\):
1. \(1 + i\)  
2. \(\sqrt{3} + i\)  
3. \(-5\)  
4. \(3i\)  
5. \(-2i\)  
6. \(-1 - \sqrt{3}i\)  
7. \(1 - i\)  
8. \(2 - 2i\)  
9. \(-\sqrt{2} - \sqrt{2}i\)  
10. \(-3 + \sqrt{3}i\)  
11. \(\frac{1}{2} - \frac{\sqrt{3}}{2}i\)  
12. \(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\)

Express each of the following in the form \(a + bi\):
13. \(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\)  
14. \(\sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)\)  
15. \(6 \left( \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right)\)  
16. \(2\sqrt{2} \left( \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)\)  
17. \(10 \left( \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right)\)  
18. \(2 \left( \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} \right)\)
Express each of the following complex numbers in the form $r(\cos \theta + i \sin \theta)$, where $i^2 = -1$:

19. $(1+i)^2$
20. $\frac{2}{1-i}$
21. $\frac{2}{\sqrt{3}+i}$
22. $\frac{1}{(1-i)^2}$

23. $z_1 = 2(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})$ and $z_2 = 5\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)$, where $i^2 = -1$.
   Calculate $z_1 z_2$ in the form $x+yi$, where $x, y \in \mathbb{R}$.

24. $z_1 = \sqrt{2}\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)$ and $z_2 = 4\left(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}\right)$, where $i^2 = -1$.
   Calculate $z_1 z_2$ in the form $x+yi$, where $x, y \in \mathbb{R}$.

25. $z_1 = 4\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$ and $z_2 = 2\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)$, where $i^2 = -1$.
   Calculate $\sqrt{z_1 z_2}$.

26. $z_1 = 2\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)$ and $z_2 = \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)$.
   Calculate $\frac{z_1}{z_2}$ in the form $x+yi$, where $x, y \in \mathbb{R}$.

27. $(a+bi)(2+5i) = 7+3i$. Express $a+bi$ in the form $r \cos \theta + i \sin \theta$.

**De Moivre’s Theorem**

\[[r(\cos \theta + i \sin \theta)]^n = r^n(\cos n\theta + i \sin n\theta)], \text{ where } n \in \mathbb{Q}.

There are three applications of De Moivre’s Theorem

1. Finding powers of complex numbers
2. Proving trigonometric identities
3. Finding roots of complex numbers.

**1. Finding Powers of Complex Numbers**

**Method:**

1. Write the number in polar form.
2. Apply De Moivre’s Theorem.
3. Simplify the result.

Note: After applying De Moivre’s Theorem the angle can be very large. However, we can keep subtracting $2\pi$ until the angle is in the range $0 < \theta < 2\pi$. 

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Example

Express \((-1 + i)^{10}\) in the form \(x + yi\). \(x, y \in \mathbb{R}\) and \(i^2 = -1\).

Solution:

1. \(r = |-1 + i| = \sqrt{(-1)^2 + 1^2} = \sqrt{1 + 1} = \sqrt{2}\)
   \[\tan \alpha = \frac{1}{1} = 1 \Rightarrow \alpha = \frac{\pi}{4}\]
   \[\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}\]
   \[\therefore (-1 + i) = \sqrt{2} \left( \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)\] (rectangular form to polar form)

2. \[(-1 + i)^{10} = \left[ \sqrt{2} \left( \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right) \right]^{10}\] (raise both sides to the power of 10)
   \[= (\sqrt{2})^{10} \left[ \cos 10 \left( \frac{3\pi}{4} \right) + i \sin 10 \left( \frac{3\pi}{4} \right) \right]\] (apply De Moivre’s Theorem)
   \[= 32 \left( \cos \frac{30\pi}{4} + i \sin \frac{30\pi}{4} \right)\]
   \[= 32 \left( \cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} \right)\]
   \[= 32(0 - i) = 0 - 32i\]
   \[\therefore (-1 + i)^{10} = 0 - 32i\]

Exercise 2.9

Use De Moivre’s Theorem to write each of the following in the form \(a + bi\):

1. \(\left( \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)^6\)
   2. \(\left( \cos \frac{\pi}{5} + i \sin \frac{\pi}{5} \right)^5\)
   3. \(\left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)^{10}\)
   4. \(\left( \cos \frac{\pi}{8} + i \sin \frac{\pi}{8} \right)^{12}\)
   5. \(\left[2 \left( \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) \right]^6\)
   6. \(\left[ \sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \right]^6\)
   7. \(\left( \cos \frac{\pi}{12} + i \sin \frac{\pi}{12} \right)^8\)
   8. \(\left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right)^7\)

9. Express \(\sqrt{3} + i\) in the form \(r(\cos \theta + i \sin \theta)\), where \(i^2 = -1\).

   Hence, use De Moivre’s Theorem to express \((\sqrt{3} + i)^3\) in the form \(a + bi\).

Use De Moivre’s Theorem to write each of the following in the form \(x + yi\):

10. \((1 + i)^8\)
    11. \((-1 + i)^4\)
    12. \((\sqrt{3} - i)^3\)
    13. \((-2 - 2i)^5\)
14. \((\sqrt{2} - \sqrt{2}i)^6\)
15. \((2 - 2\sqrt{3}i)^4\)
16. \((\frac{\sqrt{3}}{2} + \frac{1}{2}i)^9\)
17. \((\frac{1}{2} - \frac{\sqrt{3}}{2}i)^{20}\)

18. Show that: \(2(1 - i)^4 = (1 + \sqrt{3}i)^3\).

19. Evaluate: \(\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)^4 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)^2\).

20. Express \(\frac{1 + 3i}{2 + i}\) in the form \(r(\cos \theta + i \sin \theta)\), where \(i^2 = -1\).

Hence, evaluate \(\left(\frac{1 + 3i}{2 + i}\right)^{10}\).

21. \(z = \frac{1 + \sqrt{3}i}{2}\). Show that \(z^{13} = z\).

22. (i) Express \(\frac{\sqrt{3} + i}{1 + \sqrt{3}i}\) in the form \(r(\cos \theta + i \sin \theta)\), where \(i^2 = -1\).

(ii) Hence, evaluate \(\left(\frac{\sqrt{3} + i}{1 + \sqrt{3}i}\right)^6\).

2. Proving Trigonometric Identities

De Moivre’s Theorem can be used to prove trigonometric identities by expressing \(\cos n\theta\) and \(\sin n\theta\) as polynomials in \(\cos \theta\) and \(\sin \theta\), respectively.

**Example**

Using De Moivre’s Theorem, prove that: \(\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta\).

Solution:

De Moivre’s Theorem: \((\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta\).

Therefore, by De Moivre’s Theorem:

\[
\begin{align*}
\cos 3\theta + i \sin 3\theta = & \left(\cos \theta + i \sin \theta\right)^3 \\
= & \left(\binom{3}{0} \cos^3 \theta + \binom{3}{1} \cos^2 \theta \sin \theta + \binom{3}{2} \cos \theta \sin^2 \theta + \binom{3}{3} \sin^3 \theta\right) + i \left(3 \cos^3 \theta \sin \theta - 3 \cos \theta \sin^2 \theta\right) \\
= & \cos^3 \theta + 3 \cos^2 \theta \sin \theta + 3 \cos \theta \sin^2 \theta + i \sin^3 \theta \\
\end{align*}
\]

\[
\begin{align*}
\cos 3\theta + i \sin 3\theta = & \cos^3 \theta + 3 \cos^2 \theta \sin \theta - 3 \cos \theta \sin^2 \theta - i \sin^3 \theta \\
\end{align*}
\]

\((i^2 = -1, \ i^3 = -i)\)

\[
\begin{array}{cccc}
R & I & R & I \\
\end{array}
\]

Equating the real parts:

\[
\begin{align*}
\cos 3\theta = & \cos^3 \theta - 3 \cos \theta \sin^2 \theta \\
\cos 3\theta = & \cos^3 \theta - 3 \cos \theta \sin^2 \theta + \sin^2 \theta = 1 - \cos^2 \theta \\
\cos 3\theta = & \cos^3 \theta - 3 \cos \theta + 3 \cos^3 \theta \\
\cos 3\theta = & 4 \cos^3 \theta - 3 \cos \theta \\
\end{align*}
\]

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Exercise 2.10

1. Using De Moivre’s Theorem, prove that:
   (i) $\sin 2\theta = 2 \sin \theta \cos \theta$
   (ii) $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$
   Hence, express $\tan 2\theta$ in terms of $\tan \theta$.

2. Using De Moivre’s Theorem, prove that:
   (i) $\sin 3\theta = 3 \cos^2 \theta \sin \theta - \sin^3 \theta = 3 \sin \theta - 4 \sin^3 \theta$
   (ii) $\cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta = 4 \cos^3 \theta - 3 \cos \theta$
   Hence, express $\tan 3\theta$ in terms of $\tan \theta$.

3. Using De Moivre’s Theorem, prove that:
   (i) $\cos 4\theta = 8 \cos^4 \theta - 8 \cos^2 \theta + 1$
   (ii) $\sin 4\theta = 4 \cos^3 \theta \sin \theta - 4 \cos \theta \sin^3 \theta$
   Hence, express $\tan 4\theta$ in terms of $\tan \theta$.

4. Using De Moivre’s Theorem, prove that:
   (i) $\sin 5\theta = 16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta$
   (ii) $\cos 5\theta = 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta$

3. Finding Roots of Complex Numbers

From trigonometry we know that:

$$\cos \theta = \cos(\theta \pm 2\pi) = \cos(\theta \pm 4\pi) = \cos(\theta \pm 6\pi) = \cos(\theta \pm 2n\pi), \quad n \in \mathbb{Z}.$$  

$$\sin \theta = \sin(\theta \pm 2\pi) = \sin(\theta \pm 4\pi) = \sin(\theta \pm 6\pi) = \cos(\theta \pm 2n\pi), \quad n \in \mathbb{Z}.$$  

In other words, when $2\pi, 4\pi, 6\pi, \ldots$ is added to, or subtracted from, an angle, the value of sine or cosine is unchanged.

Thus, we can write:

$$r(\cos \theta + i \sin \theta) = r[\cos(\theta + 2n\pi) + i \sin(\theta + 2n\pi)], \text{ for } n \in \mathbb{Z}.$$  

When a complex number is written in the form $r[\cos(\theta + 2n\pi) + i \sin(\theta + 2n\pi)]$, the complex number is said to be written in general polar form.

Method for Finding Roots:

1. Write the number in polar form.
2. Write the number in general polar form.
3. Apply De Moivre’s Theorem.
4. Let $n = 0, 1, 2, \ldots$ (as required).
Example

Use De Moivre’s Theorem to find the three roots of the equation $z^3 - 8i = 0$.

Solution:

$$z^3 - 8i = 0$$
$$z^3 = 8i$$
$$z = 0 + 8i$$  \hspace{1cm} \text{(rectangular form)}

1. Write $0 + 8i$ in polar form.

$$0 + 8i = (0, 8)$$
$$r = 8 \text{ and } \theta = \frac{\pi}{2}$$

Polar form: $0 + 8i = 8 \left( \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)$

2. Write in general polar form.

$$8 \left[ \cos \left( \frac{\pi}{2} + 2n\pi \right) + i \sin \left( \frac{\pi}{2} + 2n\pi \right) \right]$$

$$= 8 \left[ \cos \left( \frac{\pi + 4n\pi}{2} \right) + i \sin \left( \frac{\pi + 4n\pi}{2} \right) \right]$$

(add $2n\pi$ to the angle)

$$\left( \frac{\pi}{2} + 2n\pi = \frac{\pi + 4n\pi}{2} \right)$$

3. Apply De Moivre’s Theorem.

$$z^3 = 8 \left[ \cos \left( \frac{\pi + 4n\pi}{2} \right) + i \sin \left( \frac{\pi + 4n\pi}{2} \right) \right]$$

$$z = \left[ 8 \left[ \cos \left( \frac{\pi + 4n\pi}{2} \right) + i \sin \left( \frac{\pi + 4n\pi}{2} \right) \right] \right]^{1/3}$$

(take the cube root of both sides)

$$z = 8^{1/3} \left[ \cos \left( \frac{\pi + 4n\pi}{6} \right) + i \sin \left( \frac{\pi + 4n\pi}{6} \right) \right]$$

(apply De Moivre’s Theorem)

$$z = 2 \left[ \cos \left( \frac{\pi + 4n\pi}{6} \right) + i \sin \left( \frac{\pi + 4n\pi}{6} \right) \right]$$

4. Let $n = 0, 1$ and $2$ to get the three different roots.

(Note: Letting $n = 3, 4, 5, \ldots$ merely regenerates the same roots.)

$n = 0$: $z = 2 \left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right) = 2 \left( \frac{\sqrt{3}}{2} + \frac{1}{2}i \right) = \sqrt{3} + i$

$n = 1$: $z = 2 \left( \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right) = 2 \left( -\frac{\sqrt{3}}{2} + \frac{1}{2}i \right) = -\sqrt{3} + i$

$n = 2$: $z = 2 \left( \cos \frac{9\pi}{6} + i \sin \frac{9\pi}{6} \right) = 2 \left( \cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} \right) = 2(0 - i) = -2i$

Notes:

1. The same method is used if the index is a rational number (fraction).
   For example, $(1 - \sqrt{3}i)^{\frac{3}{2}}$.

2. The number of different roots is the same as the bottom number in the fraction.
   Thus, $(1 - \sqrt{3}i)^{\frac{3}{2}}$ will have two different roots.
Example

Express \(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\) in the form \(r(\cos \theta + i \sin \theta)\) where \(i^2 = -1\).

Using De Moivre’s Theorem, find two values of \((-\frac{1}{2} + \frac{\sqrt{3}}{2}i)^{32}\).

Solution:

1. Write \((-\frac{1}{2} + \frac{\sqrt{3}}{2}i)\) in polar form.

\[
-\frac{1}{2} + \frac{\sqrt{3}}{2}i = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)
\]

\[
r = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{1} = 1
\]

\[
\tan \alpha = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} \Rightarrow \alpha = \frac{\pi}{3}
\]

\[
\therefore \theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}
\]

\[
-\frac{1}{2} + \frac{\sqrt{3}}{2}i = 1 \left(\cos \left(\frac{2\pi}{3} + 2\pi n\right) + i \sin \left(\frac{2\pi}{3} + 2\pi n\right)\right)
\]

2. Write in general polar form.

\[
= \cos \left(\frac{2\pi + 6\pi n}{3}\right) + i \sin \left(\frac{2\pi + 6\pi n}{3}\right) = \cos \left(\frac{2\pi}{3} + 2\pi n\right) + i \sin \left(\frac{2\pi}{3} + 2\pi n\right)
\]

(3) Apply De Moivre’s Theorem.

\[
\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^{32} = \cos \left(\frac{2\pi + 6\pi n}{3}\right) + i \sin \left(\frac{2\pi + 6\pi n}{3}\right)
\]

\[
\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^{32} = \cos \left(\frac{3}{2} \left(\frac{2\pi + 6\pi n}{3}\right)\right) + i \sin \left(\frac{3}{2} \left(\frac{2\pi + 6\pi n}{3}\right)\right)
\]

(4) Let \(n = 0\) and \(1\) to find the two roots.

(Note: Letting \(n = 2, 3, 4, \ldots\) merely regenerates the same roots.)

\(n = 0: \quad \cos \pi + i \sin \pi = -1 + i(0) = -1\)

\(n = 1: \quad \cos 4\pi + i \sin 4\pi = \cos 2\pi + i \sin 2\pi = 1 + i(0) = 1\)
Exercise 2.11

1. Express $-1$ in the form $r(\cos \theta + i \sin \theta)$, where $i^2 = -1$.
   Use De Moivre’s Theorem to find the three roots of the equation $z^3 = -1$.
   If the roots are $z_1$, $z_2$ and $z_3$, show that $z_1 + z_2 + z_3 = 0$.

2. Express $-8i$ in the form $r(\cos \theta + i \sin \theta)$, where $i^2 = -1$.
   Use De Moivre’s Theorem to find the three roots of the equation $z^3 + 8i = 0$.
   If the roots are $\alpha$, $\beta$ and $\gamma$, show that $\alpha + \beta + \gamma = 0$.

Use De Moivre’s Theorem to find all the solutions of each of the following equations:

3. $z^2 = -4i$

4. $z^2 = 2 - 2\sqrt{3}i$

5. $z^3 = -8$

6. $z^4 = 1$

7. $z^3 = -64i$

8. $z^3 = 27i$

9. Use De Moivre’s Theorem to find the six roots of the equation $z^6 + 64 = 0$.

10. Write $-8 - 8\sqrt{3}i$ in the form $r(\cos \theta + i \sin \theta)$, where $i^2 = -1$.
    Use De Moivre’s Theorem to find the four roots of the equation $z^4 = -8 - 8\sqrt{3}i$.

11. Write $1 - \frac{\sqrt{3}}{2} i$ in the form $r(\cos \theta + i \sin \theta)$, where $i^2 = -1$.

   Using De Moivre’s Theorem, find the two values of \( \left( \frac{1}{2} - \frac{\sqrt{3}}{2} i \right)^{3/2} \).

12. Write $2(-1 + \sqrt{3}i)$ in the form $r(\cos \theta + i \sin \theta)$, where $i^2 = -1$.
    Using De Moivre’s Theorem, find the two values of \( [2(-1 + \sqrt{3}i)]^{3/2} \).

13. Write $4i$ in the form $r(\cos \theta + i \sin \theta)$, where $i^2 = -1$.
    Using De Moivre’s Theorem, find the two values of \( (4i)^{5/2} \).
Chapter 7

Sequences and Series

Sequence

A sequence is a set of numbers, separated by commas, in which each number after the first is formed by some definite rule.

Note: Each number in the set is called a term of the sequence.

1. 5, 9, 13, 17, . . .
   Each number after the first is obtained by adding 4 to the previous number.
   In this example, 5 is called the first term, 9 the second term and so on.

2. 1, 3, 9, 27, . . .
   Each number after the first is obtained by multiplying the previous number by 3.
   In this example, 1 is called the first term, 3 the second term and so on.

General Term, \( u_n \)

The terms of a sequence can be expressed as \( u_1, u_2, u_3, u_4, \ldots \).
A sequence which follows a regular pattern can be described by a rule, or formula, called the general term. We use the symbol \( u_n \) to denote the general term. \( u_n \) may be used to generate terms of the sequence (sometimes \( T_n \) is used instead of \( u_n \)).
Consider the sequence whose general term is: \( u_n = n^2 + 3n \).
To generate any term of the sequence, put in the appropriate value for \( n \) on both sides:

\[
\begin{align*}
   u_n &= n^2 + 3n \\
   u_1 &= (1)^2 + 3(1) = 1 + 3 = 4 \\
   u_4 &= (4)^2 + 3(4) = 16 + 12 = 28 \\
   u_7 &= (7)^2 + 3(7) = 49 + 21 = 70
\end{align*}
\]

The notation \( u_n = n^2 + 3n \) is very similar to function notation where \( n \) is the input and \( u_n \) is the output, i.e. \((n, u_n)\).

Note: \( n \) used with this meaning must always be a non-negative whole number.
It can never be negative or fractional. In other words, \( n \in \mathbb{N} \).
Example

A sequence is given by \( u_n = n^2 - 3n \), where \( n \in \mathbb{N}_0 \).

(i) Find \( u_{10} \).

(ii) For what value of \( n \in \mathbb{N}_0 \) is \( u_n = 40 \)?

Solution:

(i) \( u_n = n^2 - 3n \)

\[
\begin{align*}
u_{10} &= (10)^2 - 3(10) \\
      &= 100 - 30 \\
      &= 70
\end{align*}
\]

Given: \( u_n = 40 \)

\[
\begin{align*}
&\therefore n^2 - 3n = 40 \\
&n^2 - 3n - 40 = 0 \\
&(n + 5)(n - 8) = 0 \\
&n = -5 \quad \text{or} \quad n = 8
\end{align*}
\]

Thus, \( n = 8 \), as \( n \in \mathbb{N}_0 \).

Example

If \( u_n = (n - 10)3^n \), verify that: \( u_{n+2} - 6u_{n+1} + 9u_n = 0 \).

Solution:

The basic idea is to express \( u_{n+1} \) and \( u_{n+2} \) in terms of \( n \) and \( 3^n \), the lowest power of 3, and substitute these into the given expression.

To find \( u_{n+1} \), replace \( n \) with \( (n + 1) \); to find \( u_{n+2} \), replace \( n \) with \( (n + 2) \).

\[
\begin{align*}
u_n &= (n - 10)3^n \\
u_{n+1} &= [(n + 1) - 10]3^n = (n + 1 - 10)3^n = (n - 9)3^n \\
u_{n+2} &= [(n + 2) - 10]3^{n+2} = (n + 2 - 10)3^{n+2} = n - 8)3^{n+2} = (n - 8)9 \cdot 3^n = 9(n - 8)3^n
\end{align*}
\]

\[
\begin{align*}
u_{n+2} - 6u_{n+1} + 9u_n
\end{align*}
\]

\[
\begin{align*}
&\downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow \\
&= [9(n - 8)3^n] - 6[3(n - 9)3^n] + 9[(n - 10)3^n] \\
&= 9(n - 8)3^n - 18(n - 9)3^n + 9(n - 10)3^n \\
&= 3^n[9(n - 8) - 18(n - 9) + 9(n - 10)] \quad \text{(factor out } 3^n) \\
&= 3^n[9n - 72 - 18n + 162 + 9n - 90] \\
&= 3^n[18n - 18n + 162 - 162] \\
&= 3^n[0] \\
&= 0
\end{align*}
\]
Example

If \( u_n = \frac{n}{n+1} \), show that \( u_{n+1} > u_n \).

Solution:
\[
\begin{align*}
   u_n &= \frac{n}{n+1} \quad \therefore \quad u_{n+1} = \frac{(n+1)}{(n+2)} = \frac{n+1}{n+2} \\
   u_{n+1} &> u_n \\
   \frac{n+1}{n+2} &> \frac{n}{n+1} \\
   (n+1)(n+1) &> n(n+2) \quad \left( \text{multiply both sides by } (n+2) \text{ and } (n+1); \quad (n+2) \text{ and } (n+1) \text{ are both positive as } n \in \mathbb{N}_0 \right) \\
   n^2 + 2n + 1 &> n^2 + 2n \\
   1 &> 0 \quad \text{true} \quad (\text{subtract } n^2 \text{ and } 2n \text{ from both sides}) \\
   \therefore \quad u_{n+1} &> u_n
\end{align*}
\]

Notes: If \( u_{n+1} > u_n \) for all \( n \in \mathbb{N} \), then the sequence \( u_n \) is (monotonic) increasing. If \( u_{n+1} < u_n \) for all \( n \in \mathbb{N} \), then the sequence \( u_n \) is (monotonic) decreasing.

Exercise 7.1

1. If \( u_n = 3n + 2 \), find \( u_1 \) and \( u_2 \). Show that \( u_{n+1} - u_n = 3 \).

2. (i) If \( u_n = n^2 - 3 \), find \( u_1 \), \( u_2 \) and \( u_{n+1} \).
   (ii) If \( u_{n+1} - u_n = an + b \), \( a, b \in \mathbb{R} \), find the value of \( a \) and the value of \( b \).
   (iii) If \( u_n = 222 \), find the value of \( n \), \( n \in \mathbb{N} \).

3. (i) If \( u_n = n^2 + 5n \), find \( u_1 \), \( u_2 \) and \( u_{n+1} \).
   (ii) If \( u_{n+1} - u_n = pn + q \), \( p, q \in \mathbb{R} \), find the value of \( p \) and the value of \( q \).
   (iii) If \( u_n = 66 \), find the value of \( n \), \( n \in \mathbb{N} \).
   (iv) Show that: \( u_{n+1} > u_n \).

4. \( u_n = an^2 + bn \), where \( a, b \in \mathbb{R} \). If \( u_1 = 7 \) and \( u_2 = 20 \):
   (i) find the values of \( a \) and \( b \)  \hspace{1cm} (ii) find the value of \( n \in \mathbb{N} \) if \( u_n = 64 \).

5. If \( u_n = 2^n + 1 \), find \( u_1 \), \( u_2 \) and \( u_{n+1} \). Show that \( u_{n+1} > u_n \).

6. If \( u_n = (5n - 2)3^n \), show that \( u_{n+1} - 3u_n = 5(3)^{n+1} \).

7. If \( u_n = (n + 1)5^n \), show that \( u_{n+2} - 10u_{n+1} + 25u_n = 0 \).

8. If \( u_n = \frac{1}{3}(9^n - 3^n) \), show that \( u_{n+1} = 3u_n + 2(9)^n \).

9. If \( u_n = 2^{n-1} + 2^{n-1} \), show that \( u_{n+1} - 2u_n - 2^n = 0 \).
10. If \( u_n = 3 + n(n - 1)^2 \), show that \( u_{n+1} - u_n = 3n^2 - n \).

11. If \( u_n = n^2 + 4n \), find \( u_1 \) and \( u_2 \). Simplify: \( (u_{n+2} - u_{n+1}) - (u_{n+1} - u_n) \).

12. If \( u_n = 4(n + 1)! \), show that \( u_{n+1} - nu_n = 2u_n \).

13. If \( u_n = \frac{1}{n} \), show that \( u_{n+1} < u_n \), where \( n \in \mathbb{N}_0 \).

14. If \( u_n = \frac{1}{n^2} \), show that \( u_{n+1} < u_n \), where \( n \in \mathbb{N}_0 \).

15. If \( u_n = \frac{1}{2^n} \), show that \( u_{n+1} < u_n \), where \( n \in \mathbb{N}_0 \).

16. If \( u_n = \frac{n + 3}{2n + 1} \), show that \( u_{n+1} < u_n \), where \( n \in \mathbb{N}_0 \).

17. The \( n \)th term of a sequence is given by \( u_n = a(2^n) + bn + c \), where \( a, b, c \in \mathbb{R} \).

If \( u_1 = 0 \), \( u_2 = 10 \) and \( u_3 = 26 \), find:

(i) the value of \( a, b \) and \( c \)  
(ii) \( u_4 \).

**Series and Sigma (Σ) Notation**

When the terms of a sequence are added together the sum of the terms is called a series.

For example,  

Sequence: \( 1, 4, 7, 10, \ldots \)  
Series: \( 1 + 4 + 7 + 10 + \cdots \)

A **finite series** is one which ends after a finite number of terms.

An **infinite series** is one that continues indefinitely.

The sum of the first \( n \) terms of a series is denoted by \( S_n \), where:  
\[
S_n = u_1 + u_2 + u_3 + \cdots + u_n
\]

This is an example of a finite series, as there is a finite number of terms.

The finite series \( S_n \) can be expressed more concisely using sigma (Σ) notation.

\[
S_n = u_1 + u_2 + u_3 + \cdots + u_n = \sum_{r=1}^{n} u_r
\]

Notes: The letter \( r \) (called a **dummy variable**) does not appear when \( \sum_{r=1}^{n} u_r \) is written out.

\[
\sum_{r=1}^{n} u_r = u_1 + u_2 + u_3 + \cdots + u_n.
\]

Any other letter could also have been used, for example \( \sum_{i=1}^{n} u_i \) or \( \sum_{k=1}^{n} u_k \).
\[ \sum_{r=1}^{n} u_r \] is read as:

‘the sum of \( u_r \) from \( r = 1 \) to \( r = n \)’ or ‘sigma \( u_r \) from \( r = 1 \) to \( r = n \)’.

(last value of \( r \) in the series)

\[ \sum_{r=1}^{30} u_r = u_1 + u_2 + u_3 + \cdots + u_{19} + u_{20} \]

(general term) (\( \cdots \) indicates more terms)

(first value of \( r \) in the series)

The values of \( r \) increase in steps of 1 from the first term to the last term.

\[ \sum_{r=3}^{8} u_r = u_3 + u_4 + u_5 + u_6 + u_7 + u_8 \]

i.e., start at the third term, \( u_3 \), finish at the eighth term, \( u_8 \), and add these terms.

The notation can also be used to describe an infinite series.

\[ \sum_{r=1}^{\infty} u_r = u_1 + u_2 + u_3 + \cdots + u_n + \cdots \]

(\( \cdots \) indicates that the series continues indefinitely)

In this notation, \( \infty \) indicates that there is no upper limit for \( r \).

Note: \( S_1 = u_1 \), \( S_2 = u_1 + u_2 \), \( S_3 = u_1 + u_2 + u_3 \), etc.

---

**Example**

Evaluate: (i) \[ \sum_{r=0}^{5} (2r+1) \] (ii) \[ \sum_{r=1}^{4} (-1)^{r+1} 2^r \]

**Solution:**

\[ \sum_{r=0}^{5} (2r+1) = [2(0)+1] + [2(1)+1] + [2(2)+1] + [2(3)+1] + [2(4)+1] + [2(5)+1] \]

\[ = (0+1) + (2+1) + (4+1) + (6+1) + (8+1) + (10+1) \]

\[ = 1 + 3 + 5 + 7 + 9 + 11 \]

\[ = 36 \]

\[ \sum_{r=1}^{4} (-1)^{r+1} 2^r = (-1)^{1+1}(2)^1 + (-1)^{2+1}(2)^2 + (-1)^{3+1}(2)^3 + (-1)^{4+1}(2)^4 \]

\[ = (-1)^2(2) + (-1)^3(4) + (-1)^4(8) + (-1)^5(16) \]

\[ = (1)(2) + (-1)(4) + (1)(8) + (-1)(16) \]

\[ = 2 - 4 + 8 - 16 \]

\[ = -10 \]
Notice that in the second example the series alternates between positive and negative terms.

\((-1)^k = 1\) when \(k\) is even.
\((-1)^k = -1\) when \(k\) is odd.

**Find \(u_n\) when Given \(S_n\)**

\[
S_n = u_1 + u_2 + u_3 + \cdots + u_n + u_{n-1}
\]

\[
S_{n-1} = u_1 + u_2 + u_3 + \cdots + u_{n-1}
\]

\[
S_n - S_{n-1} = u_n \quad \text{(subtracting)}
\]

If \(S_n = u_1 + u_2 + u_3 + \cdots + u_n\), then:

\[
u_n = S_n - S_{n-1}
\]

This gives us a nice method to find the general term, \(u_n\), when given \(S_n\) in terms of \(n\).

**Example**

\[
S_n = u_1 + u_2 + u_3 + \cdots + u_n
\]

If \(S_n = 2n^2 - 3n\), find an expression for \(u_n\), and hence find \(u_{10}\).

**Solution:**

\[
S_n = 2n^2 - 3n
\]

\[
S_{n-1} = 2(n-1)^2 - 3(n-1)
\]

(Replace \(n\) with \((n-1)\))

\[
= 2(n^2 - 2n + 1) - 3(n - 1)
\]

\[
= 2n^2 - 4n + 2 - 3n + 3
\]

\[
= 2n^2 - 7n + 5
\]

\[
u_n = S_n - S_{n-1}
\]

\[
= (2n^2 - 3n) - (2n^2 - 7n + 5)
\]

\[
= 2n^2 - 3n - 2n^2 + 7n - 5
\]

\[
u_n = 4n - 5
\]

Thus, \(u_{10} = 4(10) - 5 = 40 - 5 = 35\).

**Exercise 7.2**

Evaluate each of the following:

1. \(\sum_{r=1}^{6} (2r + 1)\)
2. \(\sum_{r=0}^{5} (3r - 2)\)
3. \(\sum_{r=1}^{6} r^2\)
4. \(\sum_{r=1}^{5} n(n + 1)\)
5. \(\sum_{r=1}^{4} (-1)^{r+1} r^3\)
6. \(\sum_{r=0}^{6} (-1)^r 2^r\)
7. Evaluate: (i) \( \sum_{r=2}^{5} (-1)^r(r+1)(r+3) \)  
    (ii) \( \sum_{r=3}^{7} \frac{(-1)^r}{r-1} \).

8. For a sequence, \( u_n = 2n + 5 \). Find: 
    (i) \( S_1 \)  
    (ii) \( S_4 \).

9. For a sequence, \( u_n = 3(2)^n \). Find: 
    (i) \( S_2 \)  
    (ii) \( S_3 \).

10. For a sequence, \( u_n = \frac{n}{n+1} \). Find the value of \( S_3 \).

In each of the following find \( u_n \), given \( S_n = u_1 + u_2 + u_3 + \cdots + u_n \):

11. \( S_n = n^2 + 2n \)  
    12. \( S_n = n^2 - 5n \)  
    13. \( S_n = 2n^2 + n \)

14. For the series \( S_n = u_1 + u_2 + \cdots + u_n \), \( S_n = \frac{n(n+1)}{2} \).

Find: (i) \( S_{n-1} \)  
    (ii) \( u_n \)  
    (iii) \( u_{20} \).

15. For the series \( S_n = u_1 + u_2 + \cdots + u_n \), \( S_n = 2^n \).

Find: (i) \( S_{n-1} \)  
    (ii) \( u_n \)  
    (iii) \( u_{10} \)  
    (iv) \( \sqrt{u_5} \).

16. For the series \( S_n = u_1 + u_2 + \cdots + u_n \), \( S_n = 2(2)^n + n^2 \).

Find an expression for \( u_n \) and, hence, evaluate \( u_5 \).

---

### Arithmetic Sequences and Series

Consider the sequence of numbers 2, 5, 8, 11, \ldots

Each term, after the first, can be found by adding 3 to the previous term.

This is an example of an arithmetic sequence.

---

A sequence in which each term, after the first, is found by adding a constant number is called an **arithmetic sequence**.

---

The first term of an arithmetic sequence is denoted by \( a \).

The constant number, which is added to each term, is called the **common difference** and is denoted by \( d \).

Consider the arithmetic sequence 3, 5, 7, 9, 11, \ldots

\[ a = 3 \quad \text{and} \quad d = 2 \]

Each term after the first is found by adding 2 to the previous term.

Consider the arithmetic sequence 7, 2, -3, -8, \ldots

\[ a = 7 \quad \text{and} \quad d = -5 \]

Each term after the first is found by subtracting 5 from the previous term.

In an arithmetic sequence the common difference, \( d \), between any two consecutive terms is always the same.

---

Any term - previous term = \( u_n - u_{n-1} = \) constant = \( d \).
If three terms, \( u_n, u_{n+1}, u_{n+2} \), are in arithmetic sequence, then:

\[
\frac{u_{n+2} - u_{n+1}}{u_{n+1} - u_n} = a
\]

General Term of an Arithmetic Sequence

In an arithmetic sequence \( a \) is the first term and \( d \) is the common difference. Thus, in an arithmetic sequence:

\[
\begin{align*}
    u_1 &= a \\
    u_2 &= a + d \\
    u_3 &= (a + d) + d = a + 2d \\
    u_4 &= (a + 2d) + d = a + 3d \\
    & \text{and so on.}
\end{align*}
\]

Notice that the coefficient of \( d \) is always one less than the term number. Thus, the general term of an arithmetic sequence is given by:

\[
u_n = a + (n-1)d
\]

For example: \( u_8 = a + 7d \), \( u_{10} = a + 9d \).

Note: If \( u_n = pn + q \), where \( p \) and \( q \) are constants, then the sequence is arithmetic.

Arithmetic Series

If the sequence \( u_1, u_2, u_3, \ldots, u_n \) is arithmetic, then the corresponding series, \( S_n = u_1 + u_2 + u_3 + \cdots + u_n \), is an arithmetic series.

The formula for \( S_n \) of an arithmetic series can be written in terms of the first term, \( a \), and the common difference, \( d \).

\[
S_n = \frac{n}{2} [2a + (n-1)d].
\]

To derive this result:

\[
\begin{align*}
S_n &= a + (a + d) + \cdots + [a + (n-2)d] + [a + (n-1)d] \\
S_n &= [a + (n-1)d] + [a + (n-2)d] + \cdots + [a + d] + [a] \\
2S_n &= [2a + (n-1)d] + [2a + (n-1)d] + \cdots + [2a + (n-1)d] + [2a + (n-1)d] \\
2S_n &= n[2a + (n-1)d] \\
S_n &= \frac{n}{2} [2a + (n-1)d]
\end{align*}
\]
Once we find the first term, \( a \), and the common difference, \( d \), we can answer any question about an arithmetic sequence or series.

Note: If \( S_n = pn^2 + qn \), where \( p \) and \( q \) are constants, then the series is arithmetic.

**Example**

If \( k + 2, 2k + 3, 5k - 2 \) are three consecutive terms in an arithmetic sequence, find the value of \( k \), \( k \in \mathbb{R} \).

**Solution:**

We use the fact that in an arithmetic sequence the difference between any two consecutive terms is always the same.

Thus:

\[
\begin{align*}
\frac{u_{n+2} - u_{n+1}}{u_{n+1} - u_n} &= \text{(common difference)} \\
(5k - 2) - (2k + 3) &= (2k + 3) - (k + 2) \\
5k - 2 - 2k - 3 &= 2k + 3 - k - 2 \\
3k - 5 &= k + 1 \\
2k &= 6 \\
k &= 3
\end{align*}
\]

Check: When \( k = 3 \), the terms are 5, 9, 13, which are in arithmetic sequence.

**Example**

In an arithmetic series, the sum of the first six terms is given by \( S_6 = 57 \) and the fifth term is given by \( u_5 = 14 \).

Find the first term, \( a \), and the common difference, \( d \).

**Solution:**

\[
S_n = \frac{n}{2}[2a + (n - 1)d]
\]

**Given:**

\[
\begin{align*}
S_6 &= 57 \\
\therefore \quad \frac{6}{2}(2a + 5d) &= 57 \\
3(2a + 5d) &= 57 \\
2a + 5d &= 19 \quad \text{①}
\end{align*}
\]

We now solve the simultaneous equations ① and ② to find \( a \) and \( d \).

\[
\begin{align*}
2a + 8d &= 28 \quad \text{③} \times 2 \\
-2a - 5d &= 19 \quad \text{③} \times -1 \\
3d &= 9 \\
d &= 3
\end{align*}
\]

Thus, the first term is \( a = 2 \) and the common difference is \( d = 3 \).
Example

Find the sum of the series $5 + 8 + 11 + \cdots + 65$.

Solution:

We are given $a = 5$ and $d = 3$. We need to find which term of the series is 65.

**Given:**

\[
\begin{align*}
u_n &= 65 \\
\therefore a + (n - 1)d &= 65 & \text{(we know } a \text{ and } d; \text{ find } n) \\
5 + (n - 1)(3) &= 65 & \text{(put in } a = 5 \text{ and } d = 3) \\
5 + 3n - 3 &= 65 \\
3n + 2 &= 65 \\
3n &= 63 \\
\therefore n &= 21
\end{align*}
\]

Thus, there are 21 terms in the series. We need to find $S_{21}$.

\[
\begin{align*}
S_n &= \frac{n}{2} [2a + (n - 1)d] \\
S_{21} &= \frac{21}{2} [2(5) + (20)(3)] \\
&= \frac{21}{2} (10 + 60) \\
&= \frac{21}{2} (70) \\
&= 735
\end{align*}
\]

To verify that a sequence is arithmetic, we must show the following:

\[u_n - u_{n-1} = \text{constant}.\]

To show that a sequence is not arithmetic, it is necessary only to show that the difference between any two consecutive terms is not the same. In practice, this usually involves showing that $u_3 - u_2 \neq u_2 - u_1$ or similar.

Example

(i) The $n$th term of a sequence is $u_n = 3n - 2$. Verify that the sequence is arithmetic.

(ii) The $n$th term of a sequence is $u_n = n^2 - 2n + 5$. Verify that the sequence is not arithmetic.

Solution:

(i) \[
\begin{align*}
u_n &= 3n - 2 \\
u_{n-1} &= 3(n - 1) - 2 \\
&= 3n - 3 - 2 \\
&= 3n - 5
\end{align*}
\]

(ii) \[
\begin{align*}
u_n &= n^2 - 2n + 5 \\
u_{n-1} &= (n - 1)^2 - 2(n - 1) + 5 \\
&= n^2 - 2n + 1 - 2n + 2 + 5 \\
&= n^2 - 4n + 8
\end{align*}
\]