Exercise 2.7

1. If $1+2i$ is a root of $z^3+2z^2-3z+20=0$, find the other two roots.

2. If $-2+3i$ is a root of $z^3-z^2-7z-65=0$, find the other two roots.

3. Show that $z=2$ is a root of the equation $z^3-8z^2+46z-68=0$ and find the other two roots.

4. **(i)** Express in the form $a+bi$: **(a)** $(1+i)^2$ **(b)** $(1+i)^3$.
   (ii) Show that $1+i$ is a root of $z^3-5z^2+8z-6=0$ and find the other two roots.

5. Verify that $1+3i$ is a root of the equation $z^3-7z^2+20z-50=0$ and find the other roots.

6. Verify that $-2+2i$ is a root of the equation $z^3+3z^2+4z-8=0$ and find the other roots.

7. Verify that $2-i$ is a root of the equation $z^3-11z+20=0$ and find the other roots.

8. Verify that $2+3i$ is a root of the equation $2z^3-9z^2+30z-13=0$ and find the other roots.

9. Verify that i is a root of the equation $z^3-iz^2-9z+9i=0$ and find the other roots.

10. **(i)** Express in the form $a+bi$: **(a)** $(1-i)^2$ **(b)** $(1-i)^3$.
    (ii) If $1-i$ is a root of the equation $z^3-4z^2+6z+k=0$, $k\in\mathbf{R}$, find the value of k and the other roots.

11. If $1-2i$ is a root of the equation $z^3+kz^2+7z+k-2=0$, $k\in\mathbf{R}$, find the value of k and the other roots.

12. One root of the equation $z^3+az^2+bz-4=0$, $a, b\in\mathbf{R}$, is $1+i$.
    Find the value of a and the value of b.

13. One root of the equation $z^3+pz^2+z+q=0$, $p, q\in\mathbf{R}$, is $4-i$.
    Find the value of p and the value of q.

14. If $(z-2)(z^2+az+b)=z^3-4z^2+6z-4$, $a, b\in\mathbf{Z}$,
    find the value of a and the value of b.

15. Let $p(z)=z^3+(4-2i)z^2+(5-8i)z-10i$, where $i^2=-1$.
    Determine the real numbers a and b if $p(z)=(z-2i)(z^2+az+b)$.

16. **(i)** Factorise z^2-5z+6 and, hence, solve the equation $z^2-5z+6=0$.
    (ii) Show that z^2-5z+6 is a factor of $z^3+(-4+i)z^2+(1-5i)z+6(1+i)$.
    (iii) Find the three roots of the equation $z^3+(-4+i)z^2+(1-5i)z+6(1+i)=0$.

17. **(i)** Factorise z^2-4 and, hence or otherwise, solve the equation $z^2-4=0$.
    (ii) Show that z^2-4 is a factor of $z^3+(3+i)z^2-4z-4(3+i)$.
    (iii) Find the three roots of the equation $z^3+(3+i)z^2-4z-4(3+i)=0$.

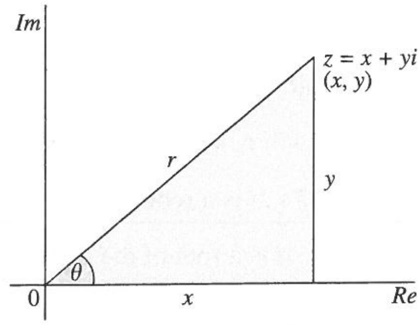
18. ki is a root of the equation $2z^3-z^2+18z-9=0$, $k\in\mathbf{R}$.
    Find the values of k and the three roots of the equation $2z^3-z^2+18z-9=0$.

Polar Coordinates and the Polar Form of a Complex Number

Polar Coordinates

Consider the complex number $z = x + yi$. The position of z on the Argand diagram can be given by Cartesian, or rectangular coordinates, (x, y). An alternative way of describing the position of z is to give its **modulus, r** and its **argument, θ**.

(r, θ) are called the polar coordinates of the complex number.



1. Modulus	**2. Argument**
The modulus $= r = \lvert z \rvert = \sqrt{x^2 + y^2}$. The modulus is the distance from the origin to the point representing the complex number on the Argand diagram.	The argument $= \theta$ (usually in radians). The argument, θ, is the angle between the positive real-axis and the line from the origin to the point $z = x + yi$.

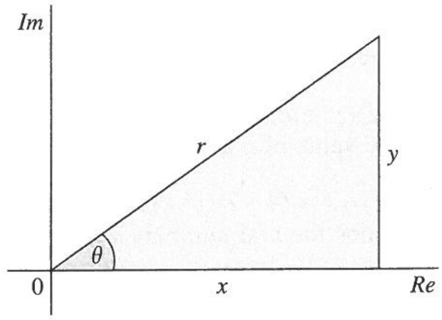
Note: Drawing a diagram can be a very good aid in calculating θ.

Polar Form

Having calculated r and θ, there is a simple connection between the Cartesian coordinates (x, y) and the polar coordinates (r, θ).

$$\frac{x}{r} = \cos\theta \qquad \frac{y}{r} = \sin\theta$$

$$x = r\cos\theta \qquad y = r\sin\theta$$



Now we can write $z = x + yi$ in terms of r and θ.

$$z = x + yi$$
$$z = (r\cos\theta) + (r\sin\theta)i$$
$$z = r(\cos\theta + i\sin\theta)$$

This is called the **polar form** of the complex number.

> The polar form of the complex number $z = x + yi$ is:
> $z = r(\cos\theta + i\sin\theta)$.

Note: It is conventional to write i before $\sin\theta$.
In other words, $i\sin\theta$ is preferable to $\sin\theta i$.

Example

Express each of these complex numbers in the form $r(\cos\theta + i\sin\theta)$, where $i^2 = -1$:

(i) $-1+i$ **(ii)** $-\sqrt{3}-i$ **(iii)** $\frac{1}{2}+\frac{\sqrt{3}}{2}i$ **(iv)** $-6i$.

Solution:

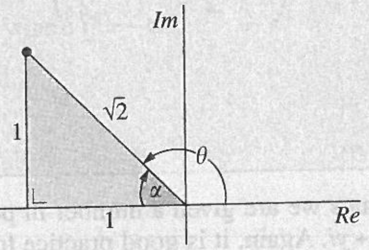
(i) $-1+i = (-1, 1)$

$r = |-1+i| = \sqrt{(-1)^2+(1)^2} = \sqrt{1+1} = \sqrt{2}$

$\tan\alpha = \frac{1}{1} = 1 \quad \Rightarrow \quad \alpha = \frac{\pi}{4}$

$\therefore \quad \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$\therefore \quad -1+i = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$



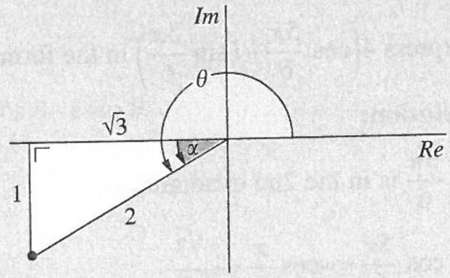
(ii) $-\sqrt{3}-i = (-\sqrt{3}, -1)$

$r = |-\sqrt{3}-i| = \sqrt{(-\sqrt{3})^2+(-1)^2} = \sqrt{3+1} = \sqrt{4} = 2$

$\tan\alpha = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad \alpha = \frac{\pi}{6}$

$\therefore \quad \theta = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$

$\therefore \quad -\sqrt{3}-i = 2\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$

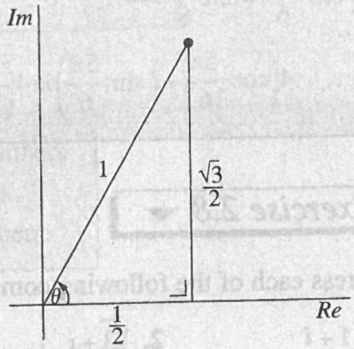


(iii) $\frac{1}{2}+\frac{\sqrt{3}}{2}i = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$r = \left|\frac{1}{2}+\frac{\sqrt{3}}{2}\right| = \sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4}+\frac{3}{4}} = \sqrt{1} = 1$

$\tan\theta = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3} \quad \Rightarrow \quad \theta = \frac{\pi}{3}$

$\therefore \quad \frac{1}{2}+\frac{\sqrt{3}}{2}i = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

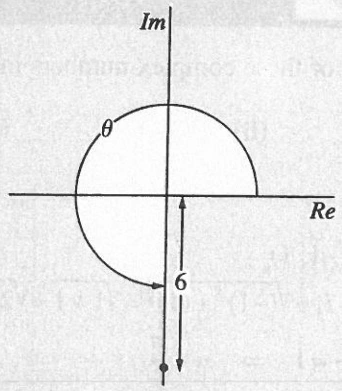


(iv) $-6i = 0 - 6i = (0, -6)$

$r = |0 - 6i| = \sqrt{0^2 + (-6)^2} = \sqrt{0 + 36} = \sqrt{36} = 6$

$\theta = \dfrac{3\pi}{2}$

$\therefore \quad -6i = 6\left(\cos \dfrac{3\pi}{2} + i \sin \dfrac{3\pi}{2}\right)$



Sometimes we are given a number in polar form $r(\cos \theta + i \sin \theta)$, and asked to write it in Cartesian form, $x + yi$. Again, it is good practice to draw a diagram.

Example

Express $4\left(\cos \dfrac{5\pi}{6} + i \sin \dfrac{5\pi}{6}\right)$ in the form $x + yi$.

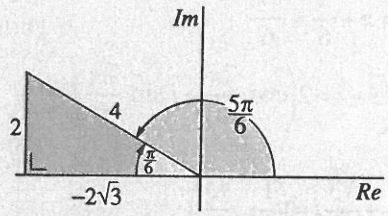
Solution:

$\dfrac{5\pi}{6}$ is in the 2nd quadrant, so:

$\cos \dfrac{5\pi}{6} = -\cos \dfrac{\pi}{6} = -\dfrac{\sqrt{3}}{2}$

$\sin \dfrac{5\pi}{6} = \sin \dfrac{\pi}{6} = \dfrac{1}{2}$

$\therefore \quad 4\left(\cos \dfrac{5\pi}{6} + i \sin \dfrac{5\pi}{6}\right) = 4\left(-\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i\right) = -2\sqrt{3} + 2i$



Exercise 2.8

Express each of the following complex numbers in the form $r(\cos \theta + i \sin \theta)$, where $i^2 = -1$:

1. $1 + i$ **2.** $\sqrt{3} + i$ **3.** -5 **4.** $3i$

5. $-2i$ **6.** $-1 - \sqrt{3}i$ **7.** $1 - i$ **8.** $2 - 2i$

9. $-\sqrt{2} - \sqrt{2}i$ **10.** $-3 + \sqrt{3}i$ **11.** $\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i$ **12.** $-\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i$

Express each of the following in the form $a + bi$:

13. $\cos \dfrac{\pi}{2} + i \sin \dfrac{\pi}{2}$ **14.** $\sqrt{2}\left(\cos \dfrac{\pi}{4} + i \sin \dfrac{\pi}{4}\right)$ **15.** $6\left(\cos \dfrac{2\pi}{3} + i \sin \dfrac{2\pi}{3}\right)$

16. $2\sqrt{2}\left(\cos \dfrac{3\pi}{4} + i \sin \dfrac{3\pi}{4}\right)$ **17.** $10\left(\cos \dfrac{5\pi}{6} + i \sin \dfrac{5\pi}{6}\right)$ **18.** $2\left(\cos \dfrac{4\pi}{3} + i \sin \dfrac{4\pi}{3}\right)$

Express each of the following complex numbers in the form $r(\cos\theta + i\sin\theta)$, where $i^2 = -1$:

19. $(1+i)^2$ **20.** $\dfrac{2}{1-i}$ **21.** $\dfrac{2}{\sqrt{3}+i}$ **22.** $\dfrac{1}{(1-i)^2}$

23. $z_1 = 2(\cos\pi + i\sin\pi)$ and $z_2 = 5\left(\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right)$, where $i^2 = -1$.

Calculate z_1z_2 in the form $x + yi$, where $x, y \in \mathbf{R}$.

24. $z_1 = \sqrt{2}\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$ and $z_2 = 4\left(\cos\dfrac{3\pi}{2} + i\sin\dfrac{3\pi}{2}\right)$, where $i^2 = -1$.

Calculate z_1z_2 in the form $x + yi$, where $x, y \in \mathbf{R}$.

25. $z_1 = 4\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$ and $z_2 = 2\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)$, where $i^2 = -1$.

Calculate $\sqrt{z_1z_2}$.

26. $z_1 = 2\left(\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right)$ and $z_2 = \left(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right)$.

Calculate $\dfrac{z_1}{z_2}$ in the form $x + yi$, where $x, y \in \mathbf{R}$.

27. $(a+bi)(2+5i) = 7+3i$. Express $a + bi$ in the form $r\cos\theta + i\sin\theta$.

De Moivre's Theorem

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)], \text{ where } n \in \mathbf{Q}.$$

There are three applications of De Moivre's Theorem

1. Finding powers of complex numbers
2. Proving trigonometric identities
3. Finding roots of complex numbers.

I. Finding Powers of Complex Numbers

Method:

1. Write the number in polar form.
2. Apply De Moivre's Theorem.
3. Simplify the result.

Note: After applying De Moivre's Theorem the angle can be very large. However, we can keep subtracting 2π until the angle is in the range $0 \leqslant \theta \leqslant 2\pi$.

Example

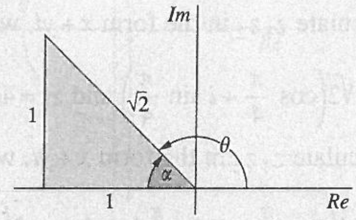
Express $(-1+i)^{10}$ in the form $x+yi$, $\quad x, y \in \mathbf{R}$ and $i^2=-1$.

Solution:

1. $r=|-1+i|=\sqrt{(-1)^2+1^2}=\sqrt{1+1}=\sqrt{2}$

$$\tan\alpha=\frac{1}{1}=1 \quad \Rightarrow \quad \alpha=\frac{\pi}{4}$$

$$\theta=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$$



$\therefore \quad (-1+i)=\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$ (rectangular form to polar form)

2. $\therefore \quad (-1+i)^{10}=\left[\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)\right]^{10}$ (raise both sides to the power of 10)

$=(\sqrt{2})^{10}\left[\cos 10\left(\frac{3\pi}{4}\right)+i\sin 10\left(\frac{3\pi}{4}\right)\right]$ (apply De Moivre's Theorem)

3. $=32\left(\cos\frac{30\pi}{4}+i\sin\frac{30\pi}{4}\right)$ $\left((\sqrt{2})^{10}=(2^{1/2})^{10}=2^5=32\right)$

$=32\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)$ $\left(\frac{30}{4}\pi=7\tfrac{1}{2}\pi=\tfrac{3}{2}\pi+6\pi=\tfrac{3}{2}\pi+3(2\pi)\right)$

$=32(0-i)=0-32i$

$\therefore \quad (-1+i)^{10}=0-32i$

Exercise 2.9

Use De Moivre's Theorem to write each of the following in the form $a+bi$:

1. $\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^6$

2. $\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)^5$

3. $\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^{10}$

4. $\left(\cos\frac{\pi}{8}+i\sin\frac{\pi}{8}\right)^{12}$

5. $\left[2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\right]^6$

6. $\left[\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]^6$

7. $\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)^8$

8. $\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^7$

9. Express $\sqrt{3}+i$ in the form $r(\cos\theta+i\sin\theta)$, where $i^2=-1$.
Hence, use De Moivre's Theorem to express $(\sqrt{3}+i)^3$ in the form $a+bi$.

Use De Moivre's Theorem to write each of the following in the form $x+yi$:

10. $(1+i)^8$ **11.** $(-1+i)^4$ **12.** $(-\sqrt{3}-i)^3$ **13.** $(-2-2i)^5$

14. $(\sqrt{2}-\sqrt{2}i)^6$ **15.** $(2-2\sqrt{3}i)^4$ **16.** $\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)^9$ **17.** $\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^{20}$

18. Show that: $2(1-i)^4=(1+\sqrt{3}i)^3$.

19. Evaluate: $\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)^4\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^2$.

20. Express $\frac{1+3i}{2+i}$ in the form $r(\cos\theta+i\sin\theta)$, where $i^2=-1$.

Hence, evaluate $\left(\frac{1+3i}{2+i}\right)^{10}$.

21. $z=\frac{1+\sqrt{3}i}{2}$. Show that $z^{13}=z$.

22. **(i)** Express $\frac{\sqrt{3}+i}{1+\sqrt{3}i}$ in the form $r(\cos\theta+i\sin\theta)$, where $i^2=-1$.

(ii) Hence, evaluate $\left(\frac{\sqrt{3}+i}{1+\sqrt{3}i}\right)^6$.

2. Proving Trigonometric Identities

De Moivre's Theorem can be used to prove trigonometric identities by expressing $\cos n\theta$ and $\sin n\theta$ as polynomials in $\cos\theta$ and $\sin\theta$, respectively.

Example

Using De Moivre's Theorem, prove that: $\cos 3\theta=4\cos^3\theta-3\cos\theta$.

Solution:

De Moivre's Theorem: $(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$.

Therefore, by De Moivre's Theorem:

$$\cos 3\theta+i\sin 3\theta=(\cos\theta+i\sin\theta)^3 \qquad \text{(put in 3 for } n \text{ on both sides)}$$

$$=\binom{3}{0}\cos^3\theta+\binom{3}{1}\cos^2\theta(i\sin\theta)+\binom{3}{2}\cos\theta(i\sin\theta)^2+\binom{3}{3}(i\sin\theta)^3$$

$$=\cos^3\theta+3\cos^2\theta(i\sin\theta)+3\cos\theta(i^2\sin^2\theta)+i^3\sin^3\theta$$

$$\underset{R}{\cos 3\theta}+\underset{I}{i\sin 3\theta}=\underset{R}{\cos^3\theta}+\underset{I}{i\,3\cos^2\theta\sin\theta}\underset{R}{-3\cos\theta\sin^2\theta}\underset{I}{-i\sin^3\theta} \qquad (i^2=-1,\ i^3=-i)$$

Equating the real parts:

$\cos 3\theta=\cos^3\theta-3\cos\theta\sin^2\theta$

$\cos 3\theta=\cos^3\theta-3\cos\theta(1-\cos^2\theta)$ $\qquad(\sin^2\theta=1-\cos^2\theta)$

$\cos 3\theta=\cos^3\theta-3\cos\theta+3\cos^3\theta$

$\cos 3\theta=4\cos^3\theta-3\cos\theta$

Exercise 2.10

1. Using De Moivre's Theorem, prove that:
   (i) $\sin 2\theta = 2 \sin\theta \cos\theta$ **(ii)** $\cos 2\theta = \cos^2\theta - \sin^2\theta$.
   Hence, express $\tan 2\theta$ in terms of $\tan\theta$.

2. Using De Moivre's Theorem, prove that:
   (i) $\sin 3\theta = 3\cos^2\theta \sin\theta - \sin^3\theta = 3\sin\theta - 4\sin^3\theta$
   (ii) $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta = 4\cos^3\theta - 3\cos\theta$.
   Hence, express $\tan 3\theta$ in terms of $\tan\theta$.

3. Using De Moivre's Theorem, prove that:
   (i) $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$
   (ii) $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$.
   Hence, express $\tan 4\theta$ in terms of $\tan\theta$.

4. Using De Moivre's Theorem, prove that:
   (i) $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$
   (ii) $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

3. Finding Roots of Complex Numbers

From trigonometry we know that:

$\cos\theta = \cos(\theta \pm 2\pi) = \cos(\theta \pm 4\pi) = \cos(\theta \pm 6\pi) = \cos(\theta \pm 2n\pi), \quad n \in \mathbf{Z}$.
$\sin\theta = \sin(\theta \pm 2\pi) = \cos(\theta \pm 4\pi) = \cos(\theta \pm 6\pi) = \cos(\theta \pm 2n\pi), \quad n \in \mathbf{Z}$.

In other words, when 2π, 4π, 6π, ... is added to, or subtracted from, an angle, the value of sine or cosine is unchanged.
Thus, we can write:

$r(\cos\theta + i\sin\theta) = r[\cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi)]$, for $n \in \mathbf{Z}$.

When a complex number is written in the form $r[\cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi)]$, the complex number is said to be written in **general polar form**.

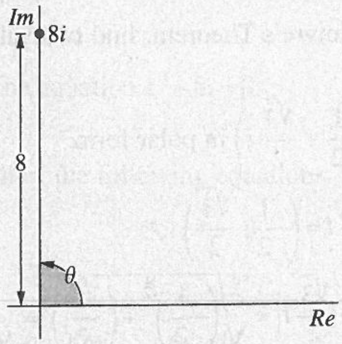
Method for Finding Roots:

1. Write the number in polar form.
2. Write the number in general polar form.
3. Apply De Moivre's Theorem.
4. Let $n = 0, 1, 2, \ldots$ (as required).

Example

Use De Moivre's Theorem to find the three roots of the equation $z^3 - 8i = 0$.

Solution:



$$z^3 - 8i = 0$$
$$z^3 = 8i$$
$$z^3 = 0 + 8i \quad \text{(rectangular form)}$$

1. Write $0 + 8i$ in polar form.

$0 + 8i = (0, 8)$

$r = 8 \quad \text{and} \quad \theta = \dfrac{\pi}{2}$

Polar form: $0 + 8i = 8\left(\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right)$

2. Write in general polar form.

$$8\left[\cos\left(\frac{\pi}{2} + 2n\pi\right) + i\sin\left(\frac{\pi}{2} + 2n\pi\right)\right] \quad \text{(add } 2n\pi \text{ to the angle)}$$
$$= 8\left[\cos\left(\frac{\pi + 4n\pi}{2}\right) + i\sin\left(\frac{\pi + 4n\pi}{2}\right)\right] \quad \left(\frac{\pi}{2} + 2n\pi = \frac{\pi + 4n\pi}{2}\right)$$

3. Apply De Moivre's Theorem.

$$z^3 = 8\left[\cos\left(\frac{\pi + 4n\pi}{2}\right) + i\sin\left(\frac{\pi + 4n\pi}{2}\right)\right]$$
$$z = \left[8\left[\cos\left(\frac{\pi + 4n\pi}{2}\right) + i\sin\left(\frac{\pi + 4n\pi}{2}\right)\right]\right]^{1/3} \quad \text{(take the cube root of both sides)}$$
$$z = 8^{1/3}\left[\cos\frac{1}{3}\left(\frac{\pi + 4n\pi}{2}\right) + i\sin\frac{1}{3}\left(\frac{\pi + 4n\pi}{2}\right)\right] \quad \text{(apply De Moivre's Theorem)}$$
$$z = 2\left[\cos\left(\frac{\pi + 4n\pi}{6}\right) + i\sin\left(\frac{\pi + 4n\pi}{6}\right)\right]$$

4. Let $n = 0$, 1 and 2 to get the three different roots.
(**Note:** Letting $n = 3, 4, 5, \ldots$ merely regenerates the same roots.)

$$n = 0: \quad z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \sqrt{3} + i$$

$$n = 1: \quad z = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = 2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = -\sqrt{3} + i$$

$$n = 2: \quad z = 2\left(\cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6}\right) = 2\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = 2(0 - i) = -2i$$

Notes: **1.** The same method is used if the index is a rational number (fraction).
For example, $(1 - \sqrt{3}i)^{3/2}$.

2. The number of different roots is the same as the bottom number in the fraction.
Thus, $(1 - \sqrt{3}i)^{3/2}$ will have two different roots.

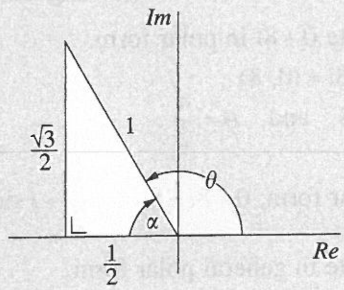
Example

Express $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$ in the form $r(\cos\theta+i\sin\theta)$ where $i^2=-1$.

Using De Moivre's Theorem, find two values of $\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^{3/2}$.

Solution:

1. Write $\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$ in polar form.



$$-\frac{1}{2}+\frac{\sqrt{3}}{2}i=\left(-\frac{1}{2},\ \frac{\sqrt{3}}{2}\right)$$

$$r=\left|-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right|=\sqrt{\left(-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\sqrt{\frac{1}{4}+\frac{3}{4}}=\sqrt{1}=1$$

$$\tan\alpha=\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}=\sqrt{3}\quad\Rightarrow\quad\alpha=\frac{\pi}{3}$$

$$\therefore\ \theta=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$$

$$\therefore\ -\frac{1}{2}+\frac{\sqrt{3}}{2}i=1\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$$

2. Write in general polar form.

$$\cos\left(\frac{2\pi}{3}+2n\pi\right)+i\sin\left(\frac{2\pi}{3}+2n\pi\right)\qquad\text{(add } 2n\pi \text{ to the angle)}$$

$$=\cos\left(\frac{2\pi+6n\pi}{3}\right)+i\sin\left(\frac{2\pi+6n\pi}{3}\right)\qquad\left(\frac{2\pi}{3}+2n\pi=\frac{2\pi+6n\pi}{3}\right)$$

3. Apply De Moivre's Theorem.

$$\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=\cos\left(\frac{2\pi+6n\pi}{3}\right)+i\sin\left(\frac{2\pi+6n\pi}{3}\right)$$

$$\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^{3/2}=\left[\cos\left(\frac{2\pi+6n\pi}{3}\right)+i\sin\left(\frac{2\pi+6n\pi}{3}\right)\right]^{3/2}\qquad\left(\text{raise both sides to the power of } \tfrac{3}{2}\right)$$

$$\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^{3/2}=\cos\frac{3}{2}\left(\frac{2\pi+6n\pi}{3}\right)+i\sin\frac{3}{2}\left(\frac{2\pi+6n\pi}{3}\right)\qquad\text{(apply De Moivre's Theorem)}$$

$$\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^{3/2}=\cos(\pi+3n\pi)+i\sin(\pi+3n\pi)$$

4. Let $n=0$ and 1 to find the two roots.

(**Note:** Letting $n=2, 3, 4, \ldots$ merely regenerates the same roots.)

$n=0$: $\quad\cos\pi+i\sin\pi=-1+i(0)=-1$

$n=1$: $\quad\cos 4\pi+i\sin 4\pi=\cos 2\pi+i\sin 2\pi=1+i(0)=1$

Exercise 2.11

1. Express -1 in the form $r(\cos\theta + i\sin\theta)$, where $i^2 = -1$.
   Use De Moivre's Theorem to find the three roots of the equation $z^3 = -1$.
   If the roots are z_1, z_2 and z_3, show that $z_1 + z_2 + z_3 = 0$.

2. Express $-8i$ in the form $r(\cos\theta + i\sin\theta)$, where $i^2 = -1$.
   Use De Moivre's Theorem to find the three roots of the equation $z^3 + 8i = 0$.
   If the roots are α, β and γ, show that $\alpha + \beta + \gamma = 0$.

Use De Moivre's Theorem to find all the solutions of each of the following equations:

3. $z^2 = -4i$
4. $z^2 = 2 - 2\sqrt{3}i$
5. $z^3 = -8$
6. $z^4 = 1$
7. $z^3 = -64i$
8. $z^3 = 27i$

9. Use De Moivre's Theorem to find the six roots of the equation $z^6 + 64 = 0$.

10. Write $-8 - 8\sqrt{3}i$ in the form $r(\cos\theta + i\sin\theta)$, where $i^2 = -1$.
    Use De Moivre's Theorem to find the four roots of the equation $z^4 = -8 - 8\sqrt{3}i$.

11. Write $\frac{1}{2} - \frac{\sqrt{3}}{2}i$ in the form $r(\cos\theta + i\sin\theta)$, where $i^2 = -1$.
    Using De Moivre's Theorem, find the two values of $\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^{3/2}$.

12. Write $2(-1 + \sqrt{3}i)$ in the form $r(\cos\theta + i\sin\theta)$, where $i^2 = -1$.
    Using De Moivre's Theorem, find the two values of $[2(-1 + \sqrt{3}i)]^{3/2}$.

13. Write $4i$ in the form $r(\cos\theta + i\sin\theta)$, where $i^2 = -1$.
    Using De Moivre's Theorem, find the two values of $(4i)^{5/2}$.

CHAPTER 7

SEQUENCES AND SERIES

Sequence

A sequence is a set of numbers, separated by commas, in which each number after the first is formed by some definite rule.

Note: Each number in the set is called a **term** of the sequence.

1. 5, 9, 13, 17, . . .
Each number after the first is obtained by adding 4 to the previous number.
In this example, 5 is called the **first term**, 9 the **second term** and so on.

2. 1, 3, 9, 27, . . .
Each number after the first is obtained by multiplying the previous number by 3.
In this example, 1 is called the **first term**, 3 the **second term** and so on.

General Term, u_n

The terms of a sequence can be expressed as $u_1, u_2, u_3, u_4, \dots$
A sequence which follows a regular pattern can be described by a rule, or formula, called the **general term**. We use the symbol u_n to denote the general term. u_n may be used to generate terms of the sequence (sometimes T_n is used instead of u_n).
Consider the sequence whose general term is: $u_n = n^2 + 3n$.
To generate any term of the sequence, put in the appropriate value for n on both sides:

$u_n = n^2 + 3n$	(general term)
$u_1 = (1)^2 + 3(1) = 1 + 3 = 4$	(first term, put in $n = 1$ on both sides)
$u_4 = (4)^2 + 3(4) = 16 + 12 = 28$	(fourth term, put in $n = 4$ on both sides)
$u_7 = (7)^2 + 3(7) = 49 + 21 = 70$	(seventh term, put in $n = 7$ on both sides)

The notation $u_n = n^2 + 3n$ is very similar to function notation where n is the input and u_n is the output, i.e. (input, output) = (n, u_n).

Note: n used with this meaning must always be a non-negative whole number.
It can never be negative or fractional. In other words, $n \in \mathbf{N}$.

Example

A sequence is given by $u_n = n^2 - 3n$, where $n \in \mathbf{N}_0$.

(i) Find u_{10}. **(ii)** For what value of $n \in \mathbf{N}_0$ is $u_n = 40$?

Solution:

(i) $u_n = n^2 - 3n$

$$u_{10} = (10)^2 - 3(10)$$
$$= 100 - 30$$
$$= 70$$

Given: $u_n = 40$

$$\therefore \quad n^2 - 3n = 40$$
$$n^2 - 3n - 40 = 0$$
$$(n+5)(n-8) = 0$$
$$n = -5 \quad \text{or} \quad n = 8$$

Thus, $n = 8$, as $n \in \mathbf{N}_0$.

Example

If $u_n = (n-10)3^n$, verify that: $u_{n+2} - 6u_{n+1} + 9u_n = 0$.

Solution:

The basic idea is to express u_{n+1} and u_{n+2} in terms of n and 3^n, the lowest power of 3, and substitute these into the given expression.

To find u_{n+1}, replace n with $(n+1)$; to find u_{n+2}, replace n with $(n+2)$.

$$u_n = (n-10)3^n$$
$$u_{n+1} = [(n+1) - 10]3^{n+1} = (n+1-10)3^1 \,.\, 3^n = (n-9)3 \,.\, 3^n = 3(n-9)3^n$$
$$u_{n+2} = [(n+2) - 10]3^{n+2} = (n+2-10)3^2 \,.\, 3^n = (n-8)9 \,.\, 3^n = 9(n-8)3^n$$

$$u_{n+2} \quad - \quad 6u_{n+1} \quad + \quad 9u_n$$
$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$
$$= [9(n-8)3^n] - 6[3(n-9)3^n] + 9[(n-10)3^n]$$
$$= 9(n-8)3^n - 18(n-9)3^n + 9(n-10)3^n$$
$$= 3^n[9(n-8) - 18(n-9) + 9(n-10)] \qquad \text{(factor out } 3^n\text{)}$$
$$= 3^n[9n - 72 - 18n + 162 + 9n - 90]$$
$$= 3^n[18n - 18n + 162 - 162]$$
$$= 3^n[0]$$
$$= 0$$

Example

If $u_n = \frac{n}{n+1}$, show that $u_{n+1} > u_n$.

Solution:

$$u_n = \frac{n}{n+1} \qquad \therefore \quad u_{n+1} = \frac{(n+1)}{(n+1)+1} = \frac{n+1}{n+2}$$

$$u_{n+1} > u_n$$

$$\frac{n+1}{n+2} > \frac{n}{n+1}$$

$$(n+1)(n+1) > n(n+2) \qquad \left(\begin{array}{l}\text{multiply both sides by } (n+2) \text{ and } (n+1);\\ (n+2) \text{ and } (n+1) \text{ are both positive as } n \in \mathbf{N}_0.\end{array}\right)$$

$$n^2 + 2n + 1 > n^2 + 2n$$

$$1 > 0 \quad \text{true} \qquad \text{(subtract } n^2 \text{ and } 2n \text{ from both sides)}$$

$$\therefore \quad u_{n+1} > u_n$$

Notes: If $u_{n+1} > u_n$, for all $n \in \mathbf{N}$, then the sequence u_n is (monotonic) increasing.
If $u_{n+1} < u_n$, for all $n \in \mathbf{N}$, then the sequence u_n is (monotonic) decreasing.

Exercise 7.1

1. If $u_n = 3n + 2$, find u_1 and u_2. Show that $u_{n+1} - u_n = 3$.
2. (i) If $u_n = n^2 - 3$, find u_1, u_2 and u_{n+1}.
   (ii) If $u_{n+1} - u_n = an + b$, $a, b \in \mathbf{R}$, find the value of a and the value of b.
   (iii) If $u_n = 222$, find the value of n, $n \in \mathbf{N}$.
3. (i) If $u_n = n^2 + 5n$, find u_1, u_2 and u_{n+1}.
   (ii) If $u_{n+1} - u_n = pn + q$, $p, q \in \mathbf{R}$, find the value of p and the value of q.
   (iii) If $u_n = 66$, find the value of n, $n \in \mathbf{N}$.
   (iv) Show that: $u_{n+1} > u_n$.
4. $u_n = an^2 + bn$, where $a, b \in \mathbf{R}$. If $u_1 = 7$ and $u_2 = 20$:
   (i) find the values of a and b (ii) find the value of $n \in \mathbf{N}$ if $u_n = 64$.
5. If $u_n = 2^n + 1$, find u_1, u_2 and u_{n+1}. Show that $u_{n+1} > u_n$.
6. If $u_n = (5n - 2)3^n$, show that $u_{n+1} - 3u_n = 5(3)^{n+1}$.
7. If $u_n = (n+1)5^n$, show that $u_{n+2} - 10u_{n+1} + 25u_n = 0$.
8. If $u_n = \frac{1}{3}(9^n - 3^n)$, show that $u_{n+1} = 3u_n + 2(9)^n$.
9. If $u_n = 2^{2n-1} + 2^{n-1}$, show that $u_{n+1} - 2u_n - 2^{2n} = 0$.

10. If $u_n = 3 + n(n-1)^2$, show that $u_{n+1} - u_n = 3n^2 - n$.

11. If $u_n = n^2 + 4n$, find u_1 and u_2. Simplify: $(u_{n+2} - u_{n+1}) - (u_{n+1} - u_n)$.

12. If $u_n = 4(n+1)!$, show that $u_{n+1} - nu_n = 2u_n$.

13. If $u_n = \frac{1}{n}$, show that $u_{n+1} < u_n$, where $n \in \mathbf{N}_0$.

14. If $u_n = \frac{1}{n^2}$, show that $u_{n+1} < u_n$, where $n \in \mathbf{N}_0$.

15. If $u_n = \frac{1}{2^n}$, show that $u_{n+1} < u_n$, where $n \in \mathbf{N}_0$.

16. If $u_n = \frac{n+3}{2n+1}$, show that $u_{n+1} < u_n$, where $n \in \mathbf{N}_0$.

17. The nth term of a sequence is given by $u_n = a(2)^n + bn + c$, where $a, b, c \in \mathbf{R}$.
If $u_1 = 0$, $u_2 = 10$ and $u_3 = 26$, find:
(i) the value of a, b and c **(ii)** u_4.

Series and Sigma (Σ) Notation

When the terms of a sequence are added together the sum of the terms is called a **series**.

For example, Sequence : $1, 4, 7, 10, \ldots$
Series : $1 + 4 + 7 + 10 + \cdots$

A **finite series** is one which ends after a finite number of terms.
An **infinite series** is one that continues indefinitely.
The sum of the first n terms of a series is denoted by S_n, where:

$$S_n = u_1 + u_2 + u_3 + \cdots + u_n$$

This is an example of a finite series, as there is a finite number of terms.
The finite series S_n can be expressed more concisely using sigma (Σ) notation.

$$\boxed{S_n = u_1 + u_2 + u_3 + \cdots + u_n = \sum_{r=1}^{n} u_r}$$

Notes: The letter r (called a **dummy variable**) does not appear when $\sum_{r=1}^{n} u_r$ is written out.
$\sum_{r=1}^{n} u_r = u_1 + u_2 + u_3 + \cdots + u_n$.
Any other letter could also have been used, for example $\sum_{i=1}^{n} u_i$ or $\sum_{k=1}^{n} u_k$.

$\sum_{r=1}^{n} u_r$ is read as:

'the sum of u_r from $r=1$ to $r=n$' or 'sigma u_r from $r=1$ to $r=n$'.

(last value of r in the series)

$$\sum_{r=1}^{20} u_r = u_1 + u_2 + u_3 + \cdots + u_{19} + u_{20}$$

(general term) (· · · indicates more terms)

(first value of r in the series)

The values of r increase in steps of 1 from the first term to the last term.

$$\sum_{r=3}^{8} u_r = u_3 + u_4 + u_5 + u_6 + u_7 + u_8$$

i.e., start at the third term, u_3, finish at the eighth term, u_8, and add these terms.

The notation can also be used to describe an infinite series.

$$\sum_{r=1}^{\infty} u_r = u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

(· · · indicates that the series continues indefinitely)

In this notation, ∞ indicates that there is no upper limit for r.

Note: $S_1 = u_1$, $S_2 = u_1 + u_2$, $S_3 = u_1 + u_2 + u_3$, etc.

Example

Evaluate: **(i)** $\sum_{r=0}^{5}(2r+1)$ **(ii)** $\sum_{r=1}^{4}(-1)^{r+1}2^r$.

Solution:

$$\begin{aligned}\sum_{r=0}^{5}(2r+1) &= [2(0)+1]+[2(1)+1]+[2(2)+1]+[2(3)+1]+[2(4)+1]+[2(5)+1]\\ &= (0+1)+(2+1)+(4+1)+(6+1)+(8+1)+(10+1)\\ &= 1+3+5+7+9+11\\ &= 36\end{aligned}$$

$$\begin{aligned}\sum_{r=1}^{4}(-1)^{r+1}2^r &= (-1)^{1+1}(2)^1+(-1)^{2+1}(2)^2+(-1)^{3+1}(2)^3+(-1)^{4+1}(2)^4\\ &= (-1)^2(2)+(-1)^3(4)+(-1)^4(8)+(-1)^5(16)\\ &= (1)(2)+(-1)(4)+(1)(8)+(-1)(16)\\ &= 2-4+8-16\\ &= -10\end{aligned}$$

Notice that in the second example the series alternates between positive and negative terms.

$(-1)^k = 1$ when k is even.

$(-1)^k = -1$ when k is odd.

Find u_n when Given S_n

$$\begin{array}{rl} S_n = & u_1 + u_2 + u_3 + \cdots + u_{n-1} + u_n \\ S_{n-1} = & u_1 + u_2 + u_3 + \cdots + u_{n-1} \\ \hline S_n - S_{n-1} = & \qquad\qquad\qquad\qquad\qquad u_n \end{array} \qquad \text{(subtracting)}$$

> If $S_n = u_1 + u_2 + u_3 + \cdots + u_n$, then:
> $$u_n = S_n - S_{n-1}$$

This gives us a nice method to find the general term, u_n, when given S_n in terms of n.

Example

$S_n = u_1 + u_2 + u_3 + \cdots + u_n$.

If $S_n = 2n^2 - 3n$, find an expression for u_n, and hence find u_{10}.

Solution:

$$S_n = 2n^2 - 3n$$

$$\begin{aligned} S_{n-1} &= 2(n-1)^2 - 3(n-1) \qquad \text{[replace } n \text{ with } (n-1)] \\ &= 2(n^2 - 2n + 1) - 3(n-1) \\ &= 2n^2 - 4n + 2 - 3n + 3 \\ &= 2n^2 - 7n + 5 \end{aligned}$$

$$\begin{aligned} u_n &= S_n - S_{n-1} \\ &= (2n^2 - 3n) - (2n^2 - 7n + 5) \\ &= 2n^2 - 3n - 2n^2 + 7n - 5 \\ u_n &= 4n - 5 \end{aligned}$$

Thus, $u_{10} = 4(10) - 5 = 40 - 5 = 35$.

Exercise 7.2

Evaluate each of the following:

1. $\sum_{r=1}^{6} (2r+1)$
2. $\sum_{r=0}^{5} (3r-2)$
3. $\sum_{r=1}^{6} r^2$
4. $\sum_{r=1}^{5} n(n+1)$
5. $\sum_{r=1}^{4} (-1)^{r+1} r^3$
6. $\sum_{r=0}^{6} (-1)^r 2^r$

7. Evaluate: (i) $\sum_{r=2}^{5}(-1)^r(r+1)(r+3)$ (ii) $\sum_{r=3}^{7}\frac{(-1)^r}{r-1}$.

8. For a sequence, $u_n = 2n+5$. Find: (i) S_1 (ii) S_4.

9. For a sequence, $u_n = 3(2)^n$. Find: (i) S_2 (ii) S_3.

10. For a sequence, $u_n = \frac{n}{n+1}$. Find the value of S_3.

In each of the following find u_n, given $S_n = u_1 + u_2 + u_3 + \cdots + u_n$:

11. $S_n = n^2 + 2n$ 12. $S_n = n^2 - 5n$ 13. $S_n = 2n^2 + n$

14. For the series $S_n = u_1 + u_2 + \cdots + u_n$, $S_n = \frac{n(n+1)}{2}$.
Find: (i) S_{n-1} (ii) u_n (iii) u_{20}.

15. For the series $S_n = u_1 + u_2 + \cdots + u_n$, $S_n = 2^n$.
Find: (i) S_{n-1} (ii) u_n (iii) u_{10} (iv) $\sqrt{u_9}$.

16. For the series $S_n = u_1 + u_2 + \cdots + u_n$, $S_n = 2(2)^n + n^2$.
Find an expression for u_n and, hence, evaluate u_8.

Arithmetic Sequences and Series

Consider the sequence of numbers 2, 5, 8, 11, . . .
Each term, after the first, can be found by adding 3 to the previous term.
This is an example of an arithmetic sequence.

> A sequence in which each term, after the first, is found by adding a constant number is called an **arithmetic sequence**.

The first term of an arithmetic sequence is denoted by *a*.
The constant number, which is added to each term, is called the **common difference** and is denoted by *d*.
Consider the arithmetic sequence 3, 5, 7, 9, 11, . . .

$$a = 3 \quad \text{and} \quad d = 2$$

Each term after the first is found by adding 2 to the previous term.

Consider the arithmetic sequence 7, 2, −3, −8, . . .

$$a = 7 \quad \text{and} \quad d = -5$$

Each term after the first is found by subtracting 5 from the previous term.
In an arithmetic sequence the common difference, *d*, between any two consecutive terms is always the same.

> Any term − previous term $= u_n - u_{n-1} =$ constant $= d$.

> If three terms, u_n, u_{n+1}, u_{n+2}, are in arithmetic sequence, then:
> $$u_{n+2} - u_{n+1} = u_{n+1} - u_n.$$

General Term of an Arithmetic Sequence

In an arithmetic sequence a is the first term and d is the common difference.
Thus, in an arithmetic sequence:

$$\begin{aligned}
u_1 &= a & &= a \\
u_2 &= a + d & &= a + d \\
u_3 &= (a + d) + d & &= a + 2d \\
u_4 &= (a + 2d) + d & &= a + 3d \quad \text{and so on.}
\end{aligned}$$

Notice that the coefficient of d is always **one less** than the term number.
Thus, the general term of an arithmetic sequence is given by:

$$\boxed{u_n = a + (n-1)d}$$

For example: $u_8 = a + 7d$, $u_{10} = a + 9d$.

Note: If $u_n = pn + q$, where p and q are constants, then the sequence is arithmetic.

Arithmetic Series

If the sequence $u_1, u_2, u_3, \ldots, u_n$ is arithmetic, then the corresponding series,

$S_n = u_1 + u_2 + u_3 + \cdots + u_n$, is an arithmetic series.

The formula for S_n of an arithmetic series can be written in terms of the first term, a, and the common difference, d.

> If $S_n = u_1 + u_2 + u_3 + \cdots + u_n$ is an arithmetic series, then:
> $$S_n = \frac{n}{2}[2a + (n-1)d].$$

To derive this result:

$$\begin{array}{rl}
S_n = & [a] + [a+d] + \cdots + [a + (n-2)d] + [a + (n-1)d] \\
S_n = & [a + (n-1)d] + [a + (n-2)d] + \cdots + [a + d] + [a] \quad \text{(in reverse)} \\
\hline
2S_n = & [2a + (n-1)d] + [2a + (n-1)d] + \cdots + [2a + (n-1)d] + [2a + (n-1)d] \quad \text{(add)} \\
2S_n = & n[2a + (n-1)d] \\
S_n = & \frac{n}{2}[2a + (n-1)d]
\end{array}$$

Once we find the first term, a, and the common difference, d, we can answer any question about an arithmetic sequence or series.

Note: If $S_n = pn^2 + qn$, where p and q are constants, then the series is arithmetic.

Example

If $k+2$, $2k+3$, $5k-2$ are three consecutive terms in an arithmetic sequence, find the value of k, $k \in \mathbf{R}$.

Solution:

We use the fact that in an arithmetic sequence the difference between any two consecutive terms is always the same.

Thus: $u_{n+2} - u_{n+1} = u_{n+1} - u_n$ (common difference)

$$(5k-2)-(2k+3) = (2k+3)-(k+2) \quad \text{(put in given values)}$$

$$5k-2-2k-3 = 2k+3-k-2$$

$$3k-5 = k+1$$

$$2k = 6$$

$$k = 3$$

Check: When $k=3$, the terms are 5, 9, 13, which are in arithmetic sequence.

Example

In an arithmetic series, the sum of the first six terms is given by $S_6 = 57$ and the fifth term is given by $u_5 = 14$.
Find the first term, a, and the common difference, d.

Solution:

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

Given: $S_6 = 57$

$$\therefore \quad \frac{6}{2}(2a+5d) = 57$$

$$3(2a+5d) = 57$$

$$2a+5d = 19 \quad ①$$

$$u_n = a+(n-1)d$$

Given: $u_5 = 14$

$$\therefore \quad a+4d = 14 \quad ②$$

We now solve the simultaneous equations ① and ② to find a and d.

$$2a+8d = 28 \quad ② \times 2$$

$$-2a-5d = -19 \quad ① \times -1$$

$$3d = 9$$

$$d = 3$$

$$a+4d = 14 \quad ②$$

$$a+4(3) = 14$$

$$a+12 = 14$$

$$a = 2$$

Thus, the first term is $a=2$ and the common difference is $d=3$.

Example

Find the sum of the series $5 + 8 + 11 + \cdots + 65$.

Solution:

We are given $a = 5$ and $d = 3$. We need to find which term of the series is 65.

Given: $u_n = 65$

$$\begin{aligned}
\therefore \quad a + (n-1)d &= 65 && \text{(we know } a \text{ and } d\text{; find } n\text{)}\\
5 + (n-1)(3) &= 65 && \text{(put in } a = 5 \text{ and } d = 3\text{)}\\
5 + 3n - 3 &= 65\\
3n + 2 &= 65\\
3n &= 63\\
n &= 21
\end{aligned}$$

Thus, there are 21 terms in the series. We need to find S_{21}.

$$\begin{aligned}
S_n &= \frac{n}{2}[2a + (n-1)d]\\
S_{21} &= \frac{21}{2}[2(5) + (20)(3)]\\
&= \frac{21}{2}(10 + 60)\\
&= \frac{21}{2}(70)\\
&= 735
\end{aligned}$$

To verify that a sequence is arithmetic, we must show the following:

$$u_n - u_{n-1} = \text{constant.}$$

To show that a sequence is **not arithmetic**, it is necessary only to show that the difference between any two consecutive terms is not the same. In practice, this usually involves showing that $u_3 - u_2 \neq u_2 - u_1$ or similar.

Example

(i) The nth term of a sequence is $u_n = 3n - 2$. Verify that the sequence is arithmetic.

(ii) The nth term of a sequence is $u_n = n^2 - 2n + 5$. Verify that the sequence is **not** arithmetic.

Solution:

(i)

$$\begin{aligned}
u_n &= 3n - 2\\
u_{n-1} &= 3(n-1) - 2\\
&= 3n - 3 - 2\\
&= 3n - 5
\end{aligned}$$

(ii)

$$\begin{aligned}
u_n &= n^2 - 2n + 5\\
u_{n-1} &= (n-1)^2 - 2(n-1) + 5\\
&= n^2 - 2n + 1 - 2n + 2 + 5\\
&= n^2 - 4n + 8
\end{aligned}$$