CHAPTER 16

BINOMIAL THEOREM

Factorials

Definition:

> The product of all the positive whole numbers from n down to 1 is called '**factorial** n' and is denoted by $n!$
> Thus, $n! = n(n-1)(n-2)\ldots 3.2.1$

For example, $5! = 5.4.3.2.1 = 120$ $\quad$ $8! = 8.7.6.5.4.3.2.1 = 40{,}320$

By definition: $0! = 1$.

Note:

$8! = 8.7! = 8.7.6!$ (and so on) $\quad$ $n! = n(n-1)! = n(n-1)(n-2)!$ (and so on)

Also,

$7.6! = 7!$ $\quad$ $(n+1)n! = (n+1)!$

When simplifying factorials, it is good practice to start with the larger factorial and work down to the smaller one.

Note: Later on we will show that the number of arrangements of all n different objects is given by $n!$

Example

(i) If $\dfrac{4!n!}{(n-1)!3n} = k$, find the value of k.

(ii) Solve: $\dfrac{(n-2)!}{n!} = \dfrac{1}{56}$

Solution:

(i)

$$\frac{4!n!}{(n-1)!3n}$$
$$= \frac{4.3.2.1.n(n-1)!}{(n-1)!3n}$$
$$= \frac{24n(n-1)!}{3n(n-1)!}$$
$$= 8$$

Thus, $k = 8$

(ii)

$$\frac{(n-2)!}{n!} = \frac{1}{56}$$
$$\frac{(n-2)!}{n(n-1)(n-2)!} = \frac{1}{56}$$
$$\frac{1}{n(n-1)} = \frac{1}{56}$$
$$n(n-1) = 56$$
$$n^2 - n - 56 = 0$$
$$(n-8)(n+7) = 0$$
$$n = 8 \quad \text{or} \quad n = -7$$

$n = -7$ is rejected as $n!$ is defined for natural numbers only.

Thus, $n = 8$.

Exercise 16.1

Evaluate each of the following:

1. $10!$ **2.** $\dfrac{8!}{6!2!}$ **3.** $\dfrac{20!}{18!}$ **4.** $\dfrac{(5!)^2}{2!4!}$

5. If $\dfrac{(n+5)!}{(n+3)!} = n^2 + an + b$, find the value of a and the value of b.

6. If $\dfrac{(n+1)!}{(n-1)!} = n^2 + pn + q$, find the value of p and the value of q.

7. If $(n+1)! + n^2(n-1)! = (an+1)n!$, find the value of a.

8. If $\dfrac{4!n!}{(n-1)!12n} = k$, find the value of k.

9. If $\dfrac{5!(n+1)!}{3!(n-1)!n(n-1)} = an$ find the value of a.

Solve each of the following:

10. $9(n-4)! = (n-3)!$ **11.** $\dfrac{(n+1)!}{(n-1)!} = 30$

12. $\dfrac{(n-2)!(n+1)!}{n!(n-1)!} = \dfrac{7}{5}$ **13.** $\dfrac{(2n-1)!}{(2n)!} = \dfrac{3!}{5!}$

Combinatorial Numbers

1.	$\dbinom{n}{r} = \dfrac{n!}{r!(n-r)!}$	(definition)
2.	$\dbinom{n}{r} = \dfrac{n(n-1)(n-2)\ \ldots\ (n-r+1)}{r!}$	(in practice)

Both give the same result; however, the second is easier to use in practical questions. For example:

1. $\dbinom{6}{2} = \dfrac{6!}{2!(6-2)!} = \dfrac{6!}{2!4!} = \dfrac{720}{2 \times 24} = 15$

2. $\dbinom{6}{2} = \dfrac{6\,.\,5}{2\,.\,1}\ \begin{matrix} \rightarrow \text{ start at 6, go down two terms} \\ \rightarrow \text{ start at 2, go down two terms} \end{matrix}$

$= 15$

Memory Aid: $\dbinom{n}{r} = \dfrac{n!}{r!(n-r)!} = \dfrac{\text{top!}}{\text{bottom!(top} - \text{bottom)!}}$

4. $\dfrac{(5!)^2}{2!4!}$

$\dbinom{n}{r}$ is pronounced 'n-c-r' or 'n-choose-r'.

Note: Later on we will show the number of ways of choosing r objects from n different objects is given by $\dbinom{n}{r}$.

Other results:

> **1.** $\dbinom{n}{0} = 1$ and $\dbinom{n}{n} = 1$
>
> **2.** $\dbinom{n}{r} = \dbinom{n}{n-r}$

Use result 2 if r is greater than $\dfrac{n}{2}$.

For example, $\dbinom{12}{9} = \dbinom{12}{12-9} = \dbinom{12}{3}$

Example

Calculate: **(i)** $\dbinom{10}{3}$ **(ii)** $\dbinom{6}{0}$ **(iii)** $\dbinom{30}{28}$

Solution:

(i) $\dbinom{10}{3} = \dfrac{10\,.\,9\,.\,8}{3\,.\,2\,.\,1} = 120$ (10 $\boxed{\text{nCr}}$ 2 $\boxed{=}$)

(ii) $\dbinom{6}{0} = 1$ (6 $\boxed{\text{nCr}}$ 0 $\boxed{=}$)

(iii) $\dbinom{30}{28} = \dbinom{30}{30-28} = \dbinom{30}{2} = \dfrac{30\,.\,29}{2\,.\,1} = 435$ (30 $\boxed{\text{nCr}}$ 28 $\boxed{=}$)

The following occur quite frequently when we have to solve equations involving combinatorial numbers.

$\dbinom{n}{1} = n$ $\qquad$ $\dbinom{n}{2} = \dfrac{n(n-1)}{2\,.\,1}$ $\qquad$ $\dbinom{n}{3} = \dfrac{n(n-1)(n-2)}{3\,.\,2\,.\,1}$

Example

Solve: **(i)** $\binom{n+1}{2} = 21, \quad n \in \mathbf{N}$ **(ii)** $4\binom{n}{2} = \binom{n+2}{3}, \quad n \in \mathbf{N}$

Solution:

(i)
$$\binom{n+1}{2} = 21, n \in \mathbf{N}.$$
$$\frac{(n+1)n}{2\,.\,1} = 21$$
$$n^2 + n = 42$$
$$n^2 + n - 42 = 0$$
$$(n+7)(n-6) = 0$$
$$n = -7 \quad \text{or} \quad n = 6$$
reject $n = -7$ as $-7 \notin \mathbf{N}$.

Thus, $n = 6$

(ii)
$$4\binom{n}{2} = \binom{n+2}{3}, n \in \mathbf{N}.$$
$$\frac{4(n)(n-1)}{2\,.\,1} = \frac{(n+2)(n+1)(n)}{3\,.\,2\,.\,1}$$
$$2(n-1) = \frac{(n+2)(n+1)}{6}$$
$$12n - 12 = n^2 + 3n + 2$$
$$n^2 - 9n + 14 = 0$$
$$(n-7)(n-2) = 0$$
$$n = 7 \quad \text{or} \quad n = 2$$

Exercise 16.2

Evaluate each of the following:

1. $\binom{5}{2}$ **2.** $\binom{7}{3}$ **3.** $\binom{8}{6}$ **4.** $\binom{6}{0}$ **5.** $\binom{4}{1}$

6. $\binom{5}{5}$ **7.** $\binom{12}{5}$ **8.** $\binom{20}{17}$ **9.** $\binom{50}{49}$ **10.** $\binom{100}{98}$

11. Verify that: **(i)** $\left[\binom{8}{2}\right]^2 = 4\binom{9}{4} + 5\binom{8}{3}$ **(ii)** $\sqrt{\binom{8}{2} - \binom{3}{2}} = 5$

Solve each of the following, $n \in \mathbf{N}_0$:

12. $\binom{n}{2} = 15$ **13.** $\binom{n}{2} = 10$ **14.** $\binom{n+1}{2} = 15$ **15.** $\binom{n+2}{2} = 6$ **16.** $\binom{n+1}{2} = 4n$

17. $\binom{n+2}{2} = 36$ **18.** $\binom{n}{3} = 5n$ **19.** $\binom{n+1}{2} = \frac{1}{2}(n^2+6)$ **20.** $\binom{n+1}{2} - \binom{n}{1} = 2$[illegible]

Use the following in questions 21 to 24:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{\text{top!}}{\text{bottom!(top} - \text{bottom)!}}$$

Prove each of the following:

21. $\binom{n}{r} = \binom{n}{n-r}$ **22.** $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$ **23.** $\binom{n}{r+1} = \frac{n-r}{r+1}\binom{n}{r}$ **24.** $r\binom{n}{r} = n\binom{n-1}{r-1}$

25. Show that $\binom{n+2}{n} = \binom{n+2}{2}$. Hence, or otherwise, solve $\binom{n+2}{n} = 45$.

Binomial Theorem

An expression with two terms, e.g. $a + b$, is called a **binomial**.

The Binomial Theorem is used to write down the expansion of a binomial to any power, e.g. $(a + b)^n$.

The expansion of $(a + b)^n$ is found as follows:

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}a^1 b^{n-1} + \binom{n}{n}a^0 b^n$$

Notes:

1. The expansion contains $(n + 1)$ terms (one more than the power).
2. The powers of a decrease by 1 in each successive term.
3. The powers of b increase by 1 in each successive term.
4. In each term the sum of the indices of a and b is n.
5. The power of b is always the same as the lower number in the combination bracket.
6. If the binomial is a difference, $(a - b)$, the signs will be alternately $+, -, +, -, +, \ldots$

The Binomial Expansion of $(1 + a)^n$ is found as follows:

$$(1+a)^n = \binom{n}{0} + \binom{n}{1}a + \binom{n}{2}a^2 + \cdots + \binom{n}{n-1}a^{n-1} + \binom{n}{n}a^n$$

This form of the Binomial Theorem can be used to expand a binomial to any power when the first term of the binomial is 1.

Example

Write out all terms in the expansion of $(a + b)^5$.

Solution:

The power is 5, thus there are 6 terms (always one more than the power).

Step 1: $ab + \quad ab + \quad ab + \quad ab + \quad ab + \quad ab$

(write down 6 pairs of the variables)

Step 2: $a^5b^0 + \quad a^4b^1 + \quad a^3b^2 + \quad a^2b^3 + \quad a^1b^4 + \quad a^0b^5$

(put in powers, starting with the highest power of a; sum of powers = 5 in each term)

Step 3: $\binom{5}{0}a^5b^0 + \binom{5}{1}a^4b^1 + \binom{5}{2}a^3b^2 + \binom{5}{3}a^2b^3 + \binom{5}{4}a^1b^4 + \binom{5}{5}a^0b^5$

(put in combinatorial numbers)

Step 4: $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

In practice the first three steps can be combined in one step.

Note: (any real number)$^0 = 1$, thus $a^0 = 1$, $b^0 = 1$, etc.

Example

Expand, using the Binomial Theorem, $(1 + 2x)^5$. Hence, expand $(1 - 2x)^5$.

Solution:

$(1 + 2x)^5$ (the power is 5, thus there are 6 terms)

$$= \binom{5}{0} + \binom{5}{1}(2x) + \binom{5}{2}(2x)^2 + \binom{5}{3}(2x)^3 + \binom{5}{4}(2x)^4 + \binom{5}{5}(2x)^5$$

$$= (1) + (5)(2x) + (10)(4x^2) + (10)(8x^3) + (5)(16x^4) + (1)(32x^5)$$

$$= 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5$$

The only difference between the expansions of $(1 + 2x)^5$ and $(1 - 2x)^5$ is that in the expansion of $(1 - 2x)^5$ the signs alternate +, –, +, –, +, –.

Thus $(1 - 2x)^5 = 1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$

Example

Expand, using the Binomial Theorem, $(x - 3y)^4$.

Solution:

$(x - 3y)^4$ (the power is 4, thus there are 5 terms)

$$= \binom{4}{0}x^4(-3y)^0 + \binom{4}{1}x^3(-3y)^1 + \binom{4}{2}x^2(-3y)^2 + \binom{4}{3}x^1(-3y)^3 + \binom{4}{4}x^0(-3y)^4$$

$$= (1)(x^4)(1) + (4)(x^3)(-3y) + (6)(x^2)(9y^2) + (4)(x)(-27y^3) + (1)(1)(81y^4)$$

$$= x^4 - 12x^3y + 54x^2y^2 - 108xy^3 + 81y^4$$

Example

Expand, using the Binomial Theorem, $\left(p + \frac{2}{p}\right)^4$.

Show that one of the terms is independent of p.

Solution:

$\left(p + \frac{2}{p}\right)^4$ (the power is 4, thus there are 5 terms)

$$= \binom{4}{0}(p)^4\left(\frac{2}{p}\right)^0 + \binom{4}{1}(p)^3\left(\frac{2}{p}\right)^1 + \binom{4}{2}(p)^2\left(\frac{2}{p}\right)^2 + \binom{4}{3}(p)^1\left(\frac{2}{p}\right)^3 + \binom{4}{4}(p)^0\left(\frac{2}{p}\right)^4$$

$$= (1)(p^4)(1) + (4)(p^3)\left(\frac{2}{p}\right) + (6)(p^2)\left(\frac{4}{p^2}\right) + (4)(p)\left(\frac{8}{p^3}\right) + (1)(1)\left(\frac{16}{p^4}\right)$$

$$= p^4 + 8p^2 + 24 + \frac{32}{p^2} + \frac{16}{p^4}$$

The third term, 24, is independent of p (in other words, the term does not contain p or the power of p is 0, i.e. p^0)

Exercise 16.3

Use the Binomial Theorem to expand each of the following:

1. $(a+b)^4$ **2.** $(x+y)^5$ **3.** $(p+q)^6$ **4.** $(x-y)^7$

5. $(1+2x)^4$ **6.** $(1-3x)^4$ **7.** $(1-2x)^6$ **8.** $(1-5x)^4$

9. $(3+y)^4$ **10.** $(2-x)^6$ **11.** $(2+3y)^4$ **12.** $(2x-3y)^5$

13. $\left(1+\frac{x}{2}\right)^6$ **14.** $\left(p+\frac{1}{p}\right)^5$ **15.** $\left(x+\frac{2}{x}\right)^8$ **16.** $\left(2a-\frac{1}{b}\right)^4$

17. $f(x)=(1+x)^4+(1-x)^4$. Express $f(x)$ as a series in ascending powers of x.

18. $(a+2b)^5+(a-2b)^5=pa^5+qa^3b^2+rab^4$. Find the values of p, q and r.

19. $\left(x+\frac{1}{x}\right)^4+\left(x-\frac{1}{x}\right)^4=px^4+q+\frac{r}{x^4}$. Find the values of p, q and r.

Write down and simplify the first three terms in the expansion of each of the following:

20. $(1+2a)^{12}$ **21.** $\left(1-\frac{a}{2}\right)^{10}$ **22.** $\left(x^2+\frac{2}{x}\right)^9$

23. Write out the binomial expansion of $\left(\frac{x}{2}+\frac{2}{x}\right)^4$.

Show that one of the terms is independent of x.

Evaluating Expansions of Sums and Differences

We can be asked to evaluate expansions such as $(1+\sqrt{5})^4$ and write the answer in the form $a+b\sqrt{5}$.

Example

Expand $(3+\sqrt{2})^5$ by the Binomial Theorem, and write your answer in the form $a+b\sqrt{2}$.

Solution:

$(3+\sqrt{2})^5$ (the power is 5, thus there are 6 terms)

$$=\binom{5}{0}(3)^5(\sqrt{2})^0+\binom{5}{1}(3)^4(\sqrt{2})^1+\binom{5}{2}(3)^3(\sqrt{2})^2+\binom{5}{3}(3)^2(\sqrt{2})^3+\binom{5}{4}(3)^1(\sqrt{2})^4+\binom{5}{5}(3)^0(\sqrt{2})^5$$

$$=(1)(243)(1)+(5)(81)(\sqrt{2})+(10)(27)(2)+(10)(9)(2\sqrt{2})+(5)(3)(4)+(1)(1)(4\sqrt{2})$$

$$=243+405\sqrt{2}+540+180\sqrt{2}+60+4\sqrt{2}$$

$$=843+589\sqrt{2}$$

$\left((\sqrt{2})^2=2,\quad (\sqrt{2})^3=2\sqrt{2},\quad (\sqrt{2})^4=4,\quad (\sqrt{2})^5=4\sqrt{2}\right)$

Exercise 16.4

Use the Binomial Theorem to evaluate each of the following, writing your answer in the form $a + b\sqrt{c}$:

1. $(1 + \sqrt{2})^4$
2. $(1 - \sqrt{3})^5$
3. $(1 + \sqrt{5})^6$
4. $(2 + \sqrt{6})^4$
5. $(1 + 3\sqrt{2})^4$
6. $(1 - 2\sqrt{3})^6$
7. $(2 + \sqrt{3})^6$
8. $(\sqrt{3} - \sqrt{2})^5$
9. $(1 + \sqrt{3})^4 + (1 - \sqrt{3})^4 = k$. Use the Binomial Theorem to find the value of k.
10. $(3 + \sqrt{2})^5 + (3 - \sqrt{2})^5 = k$. Use the Binomial Theorem to find the value of k.
11. Use the Binomial Theorem to show that $(1 + x)^6 + (1 - x)^6 = 2(1 + 15x^2 + 15x^4 + x^6)$.
    Hence, evaluate $(1 + \sqrt{2})^6 + (1 - \sqrt{2})^6$.
12. Use the Binomial Theorem to expand $(a + b)^5$.
    Hence, expand $(1 + 2x)^5 - (1 - 2x)^5$.
    Hence, write $(1 + 2\sqrt{3})^5 - (1 - 2\sqrt{3})^5$ in the form $k\sqrt{3}$, $\quad k \in \mathbf{N}$.
13. Use the Binomial Expansion to expand $(p + q)^4 + (p - q)^4$.
    Hence, or otherwise, write $(x + \sqrt{x^2 + 1})^4 + (x - \sqrt{x^2 + 1})^4$ as a polynomial in x.
    Hence, evaluate $(4 + \sqrt{17})^4 + (4 - \sqrt{17})^4$.

Unknown Coefficients or an Unknown Index

In some questions we have to find unknown coefficients or an unknown index (power).

When the index (power) is unknown we make use of the following:

$$\binom{n}{0} = 1 \qquad \binom{n}{1} = n \qquad \binom{n}{2} = \frac{n(n-1)}{2\,.\,1} \qquad \binom{n}{3} = \frac{n(n-1)(n-2)}{3\,.\,2\,.\,1}$$

Then equate the coefficients and solve the resultant equations.

Example

The first three terms in the binomial expansion of $(1 + ax)^n$ are $1 + 2x + \frac{5}{3}x^2$.

Find the value of n and the value of a.

Solution:

The first three terms are:

$$\begin{aligned}(1 + ax)^n &= \binom{n}{0} + \binom{n}{1}(ax) + \binom{n}{2}(ax)^2 \\ &= 1 + n(ax) + \frac{n(n-1)}{2\,.\,1}a^2x^2 \\ &= 1 + nax + \frac{n(n-1)}{2}a^2x^2\end{aligned}$$

$$\therefore \quad 1 + nax + \frac{n(n-1)}{2}a^2x^2 = 1 + 2x + \tfrac{5}{3}x^2$$

What we do next is equate the coefficients and solve the resultant equations.

$$na = 2 \quad ①$$

$$a = \left(\frac{2}{n}\right)$$

put this into ②

$$\frac{n(n-1)}{2}a^2 = \frac{5}{3} \quad ②$$

$$\frac{n(n-1)}{2} \cdot \left(\frac{2}{n}\right)^2 = \frac{5}{3}$$

$$\frac{n(n-1)}{2} \cdot \frac{4}{n^2} = \frac{5}{3}$$

$$\frac{2(n-1)}{n} = \frac{5}{3}$$

$$6(n-1) = 5n$$

$$6n - 6 = 5n$$

$$n = 6$$

From ①, $a = \dfrac{2}{n} = \dfrac{2}{6} = \dfrac{1}{3}$

Thus, $n = 6$ and $a = \frac{1}{3}$.

Exercise 16.5

1. The first four terms in the expansion of $(1 - x)^n$ are $1 - 6x + ax^2 + bx^3$. Find the value of n, the value of a and the value of b.

2. The first three terms in the expansion of $(1 + ax)^n$ are $1 + 20x + 150x^2$. Find the value of n and the value of a.

3. The first three terms in the expansion of $(1 + ax)^n$ are $1 + 28x + 336x^2$. Find the value of n and the value of a.

4. The first three terms in the expansion of $(1 + kx)^n$ are $1 - 18x + 135x^2$. Find the value of n and the value of k.

5. The first four terms in the expansion of $(1 + kx)^n$ are $1 + 20x + 180x^2 + ax^3$. Find the value of n, the value of k and the value of a.

6. The first three terms in the expansion of $(1 + ax)^n$ are $1 - 5x + \frac{45}{4}x^2$. Find the value of n and the value of a.

7. The first three terms in the expansion of $(1 + ax)^n$ are $1 + 8x + 30x^2$. Find the value of n and the value of a.

8. The first four terms in the expansion of $\left(1 - \dfrac{3x}{2}\right)^n$ are $1 - 24x + ax^2 + bx^3$.

   Find the value of n, the value of a and the value of b.

9. The first three terms in the expansion of $(1 + kx)^n$ are $1 + 2x + \frac{15}{8}x^2$.
Find the value of n and the value of k.

10. The first four terms in the expansion of $(1 + kx)^6$ are $1 + ax + 135x^2 + bx^3$.
Find the values of k, the values of a and the values of b.

11. The first three terms of the expansion of $\left(a - \dfrac{x}{3}\right)^6$, $a > 0$, in ascending powers of x, ar
$64 + 16bx + \frac{1}{3}bcx^2$, $\quad a, b, c \in \mathbf{R}$. Find the value of a, of b and of c.

12. In the expansion of $(1 + x)^{n+1}$ the coefficient of x^4 is $6k$ and in the expansion of $(1 + x)^{n-1}$ the coef ficient of x^2 is k. Find the value of n, $n > 2$.

Selecting a Particular Term

In many problems we require only a particular term. For example, the middle term, the fifth term or th term independent of x (no x term or the power of x is zero, i.e. x^0).

In these cases we make use of the '**general term**'.

General Term:

The general term in the binomial expansion of $(a + b)^n$ is:

$$u_{r+1} = \binom{n}{r} a^{n-r} b^r$$

(sum of powers = $n - r + r = n$)

$$u_{r+1} = \binom{n}{r} a^{n-r} b^r$$

(same)

Note: The number at the bottom of the combination bracket, r, is always **one less** than the terr number, $r + 1$.

Example

(i) Find, and simplify, the middle term in the binomial expansion of $\left(6x + \dfrac{y}{3}\right)^{10}$.

(ii) Find, and simplify, the third term in the expansion of $\left(2x - \dfrac{1}{x}\right)^8$.

Solution:

(i) $\left(6x+\dfrac{y}{3}\right)^{10}$

General term: $u_{r+1}=\dbinom{10}{r}(6x)^{10-r}\left(\dfrac{y}{3}\right)^{r}$

There are 11 terms in the expansion.
Thus, the middle term is u_6

$\therefore \quad u_6=u_{r+1}\Rightarrow r=5$

$$\therefore \quad u_6=\binom{10}{5}(6x)^{10-5}\left(\frac{y}{3}\right)^{5}$$

$$=\binom{10}{5}(6x)^{5}\left(\frac{y}{3}\right)^{5}$$

$$=(252)(7776x^5)\left(\frac{y^5}{243}\right)$$

$$=8064x^5y^5$$

(ii) $\left(2x-\dfrac{1}{x}\right)^{8}$

General term: $u_{r+1}=\dbinom{8}{r}(2x)^{8-r}\left(-\dfrac{1}{x}\right)^{r}$

$u_3=u_{r+1}\Rightarrow r=2$

$$\therefore \quad u_3=\binom{8}{2}(2x)^{8-2}\left(-\frac{1}{x}\right)^{2}$$

$$=\binom{8}{2}(2x)^{6}\left(-\frac{1}{x}\right)^{2}$$

$$=(28)(64x^6)\left(\frac{1}{x^2}\right)$$

$$=1792x^4$$

Example

Find **(i)** the general term **(ii)** the term independent of x, in the binomial expansion of $\left(2x-\dfrac{1}{x^2}\right)^{9}$.

Solution:

(i) $\left(2x-\dfrac{1}{x^2}\right)^{9}$

General term:

$$u_{r+1}=\binom{9}{r}(2x)^{9-r}\left(\frac{-1}{x^2}\right)^{r}$$

$$=\binom{9}{r}(2^{9-r})(x^{9-r})(x^{-2r})(-1)^{r}$$

$$=(-1)^{r}\binom{9}{r}2^{9-r}x^{9-3r}$$

(ii) For the term independent of x, the power of $x=0$

$\therefore \quad 9-3r=0$

$r=3$ (4th term)

Thus, the required term

$$=(-1)^{3}\binom{9}{3}2^{9-3}x^{9-9}$$

$$=(-1)(84)(2^6)(x^0)$$

$$=-5376 \qquad (x^0=1)$$

Example

(i) Write down the general term in the binomial expansion of $(2x + 5y^2)^n$.
(ii) If k is a constant and kx^3y^4 is a term in the binomial expansion of $(2x + 5y^2)^n$, find the value of n and the value of k.

Solution:

(i) $(2x + 5y^2)^n$

General term:

$$u_{r+1} = \binom{n}{r}(2x)^{n-r}(5y^2)^r$$
$$= \binom{n}{r}2^{n-r}x^{n-r}5^r y^{2r}$$
$$= \binom{n}{r}2^{n-r}5^r x^{n-r}y^{2r}$$

(ii) If kx^3y^4 is a particular term, then

$2r = 4$	and	$n - r = 3$
$r = 2$	and	$n - 2 = 3$
$r = 2$	and	$n = 5$

$$k = \binom{5}{2}2^3 5^2$$
$$= (10)(8)(25) = 2000$$

Exercise 16.6

Find the:

1. 4th term of $(1 + 2x)^6$
2. 5th term of $(a + 2b)^9$
3. 3rd term of $(x - 2y)^{12}$
4. 7th term of $(1 + x^2)^8$
5. 4th term of $(2x - \frac{1}{2})^4$
6. 5th term of $\left(x + \dfrac{1}{x}\right)^8$
7. 6th term of $\left(x - \dfrac{1}{x^2}\right)^{10}$
8. 4th term of $\left(1 + \dfrac{x}{2}\right)^{10}$
9. Write down the first three terms in the expansion of $(1 + 3x)^7$, and find their sum when $x = \frac{2}{3}$.
10. When $\left(1 - \dfrac{x}{2y}\right)^5$ is expanded in ascending powers of x, find:
    (i) the coefficient of x^4, when $y = 1$
    (ii) the sum of the first three terms when $x = 0.1$, $y = 0.02$.

Find the middle term in each of the following:

11. $(1 + 2x)^{10}$
12. $(a + 2b)^8$
13. $(p - 2q)^{10}$
14. $(a - 3b)^6$
15. $\left(2 - \dfrac{x}{2}\right)^6$
16. $\left(p^2 + \dfrac{1}{p}\right)^6$
17. $\left(x^2 + \dfrac{2}{x}\right)^8$
18. $\left(x^3 - \dfrac{1}{2x}\right)^8$

Find the term independent of x in the binomial expansion of each of the following:

19. $\left(x + \dfrac{2}{x}\right)^4$
20. $\left(x - \dfrac{1}{x}\right)^8$
21. $\left(2x + \dfrac{1}{x^2}\right)^3$
22. $\left(x^2 - \dfrac{1}{x}\right)^6$

23. $\left(x + \dfrac{1}{x^2}\right)^9$ **24.** $\left(x - \dfrac{1}{x^2}\right)^{18}$ **25.** $\left(x^4 + \dfrac{1}{x}\right)^{30}$ **26.** $\left(x + \dfrac{1}{\sqrt{x}}\right)^{15}$

27. If k is a constant and kxy^6 is a term in the expansion of $(2x + 3y^2)^n$, find the value of n and the value of k.

28. If h is a constant and hx^4y^2 is a term in the expansion of $(3x - 4y^2)^n$, find the value of n and the value of h.

29. Write down the binomial expansion of $(1 + 2x)^n$ in ascending powers of x as far as the term containing x^3.
Given that the coefficient of x^3 is twice the coefficient of x^2 and that both are positive, find the value of n.

30. Write down the first three terms in the binomial expansion, in ascending powers of x, of $(1 + ax)^n$, where $a \neq 0$ and $n \in \mathbf{N}$.
Given that the coefficient of x in this expansion is twice the coefficient of x^2,

(i) show that $n = \dfrac{a+1}{a}$

(ii) find the value of the coefficient of x^2 when $a = \frac{1}{7}$.

Sums of Binomial Coefficients

The binomial expansion

$$(1 + x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n$$

can be used to find the sums of binomial coefficients by letting $x = 1$ or $x = -1$ on both sides.

Other binomial series can also be used.

Example

(i) Prove that: $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n} = 2^n$.

Hence, prove that $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = 2^{n-1}$.

(ii) Evaluate: $\binom{12}{0} + \binom{12}{1} + \binom{12}{2} + \cdots + \binom{12}{11} + \binom{12}{12}$.

(iii) Evaluate the sum of the coefficients in the expansion of $(x + 3)^8$.

Solution:

(i) Consider the expansion of $(1+x)^n$.

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + \binom{n}{n}x^n$$

This is true for all x. Let $x = 1$ on both sides.

$$(1+1)^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n}$$

$$\therefore \quad 2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n}$$

Let $x = -1$ on both sides.

$$(1-1)^n = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \binom{n}{4} - \binom{n}{5} + \cdots$$

$$\therefore \quad 0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \binom{n}{4} - \binom{n}{5} + \cdots$$

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

$$= \frac{1}{2}\left[\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \binom{n}{4} + \binom{n}{5} + \cdots\right]$$

$$= \tfrac{1}{2}(2^n) \quad \text{[by result above]}$$

$$= 2^{n-1}$$

(ii) $\binom{12}{0} + \binom{12}{1} + \binom{12}{2} + \cdots + \binom{12}{12} = 2^{12} = 4{,}096$

(iii) $(x+3)^8 = \binom{8}{0}x^8 + \binom{8}{1}x^7(3) + \binom{8}{2}x^6(3)^2 + \cdots + \binom{8}{8}(3)^8$

Let $x = 1$ on both sides.

$$(1+3)^8 = \binom{8}{0} + \binom{8}{1}(3) + \binom{8}{2}(3)^2 + \cdots + \binom{8}{8}(3)^8 = 4^8$$

Thus, the sum of the coefficients is $4^8 = 65{,}536$.

Exercise 16.7

Use the binomial expansion $(1+x)^n$ to verify:

1. $\binom{8}{0} + \binom{8}{1} + \binom{8}{2} + \cdots + \binom{8}{7} + \binom{8}{8} = 2^8$

2. $\binom{16}{0} + \binom{16}{1} + \binom{16}{2} + \cdots + \binom{16}{15} + \binom{16}{16} = 2^{16}$

3. $\binom{16}{0}+\binom{16}{2}+\binom{16}{4}+\cdots+\binom{16}{14}+\binom{16}{16}=2^{15}$

4. $\binom{14}{1}+\binom{14}{3}+\binom{14}{5}+\cdots+\binom{14}{13}=\binom{14}{0}+\binom{14}{2}+\binom{14}{6}+\cdots+\binom{14}{14}$

5. $\binom{11}{0}-\binom{11}{1}+\binom{11}{2}-\binom{11}{3}+\cdots+\binom{11}{10}-\binom{11}{11}=0$

6. Evaluate the sum of the coefficients in the expansion of $(x+2)^7$.

7. Evaluate the sum of the coefficients in the expansion of $(x+y)^{12}$.

8. Evaluate the sum of the coefficients in the expansion of $(3x+1)^9$.