CHAPTER 1

ALGEBRA

Simplifying Algebraic Expressions and Fractions

Special Factors

1.	$a^2 - b^2 = (a - b)(a + b)$	Difference of two squares
2.	$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$	Difference of two cubes
3.	$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$	Sum of two cubes

Special Expansions

1. $(a + b)^2 = a^2 + 2ab + b^2$	**2.** $(a - b)^2 = a^2 - 2ab + b^2$
3. $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$	**4.** $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

These factors and expansions occur frequently and should be memorised.

Example

Factorise **(i)** $6a^2b - 9ab^2$ **(ii)** $25x^2 - 16y^2$ **(iii)** $8x^3 - 27y^3$ **(iv)** $1 + 1000p^3$

Solution:

(i) $6a^2b - 9ab^2$

$= 3ab(2a - 3b)$

(HCF is $3ab$)

(ii) $25x^2 - 16y^2$

$= (5x)^2 - (4y)^2$

$= (5x - 4y)(5x + 4y)$

(iii) $8x^3 - 27y^3$

$= (2x)^3 - (3y)^3$

$= (2x - 3y)[(2x)^2 + (2x)(3y) + (3y)^2]$

$= (2x - 3y)(4x^2 + 6xy + 9y^2)$

(iv) $1 + 1000p^3$

$= (1)^3 + (10p)^3$

$= (1 + 10p)[(1)^2 + (1)(10p) + (10p)^2]$

$= (1 + 10p)(1 + 10p + 100p^2)$

Multiplication and Division of Algebraic Fractions

Operations with algebraic fractions follow the same rules as in arithmetic. Before attempting to simplify when multiplying or dividing algebraic fractions, factorise where possible and divide top and bottom by common factors. The contents of a bracket should be considered as a single term.

Example

Simplify $\dfrac{4x^2 - 10x}{9x^2 + 6x} \div \dfrac{2x - 5}{3x + 2}$.

Solution:

$$\frac{4x^2-10x}{9x^2+6x} \div \frac{2x-5}{3x+2}$$

$$= \frac{4x^2-10x}{9x^2+6x} \times \frac{3x+2}{2x-5}$$ [turn the fraction we divide by upside down and multiply]

$$= \frac{2x(2x-5)(3x+2)}{3x(3x+2)(2x-5)}$$ [factorise the top and bottom]

$$= \frac{2}{3}$$ [divide top and bottom by the common factors, x, $(3x+2)$ and $(2x-5)$]

Addition and Subtraction of Algebraic Fractions

To add or subtract algebraic fractions do the following:

1. Factorise denominators (if necessary).
2. Find the L.C.M. of the denominators.
3. Express each fraction in terms of this L.C.M. and simplify.

Example

Express as one fraction in its lowest terms:

$$\frac{3}{x+5} - \frac{2}{x+3} + \frac{5x+19}{x^2+8x+15}$$

Solution:

$$\frac{3}{x+5} - \frac{2}{x+3} + \frac{5x+19}{x^2+8x+15}$$

$$= \frac{3}{(x+5)} - \frac{2}{(x+3)} + \frac{5x+19}{(x+5)(x+3)}$$ [factorise the denominators; their L.C.M. is $(x+5)(x+3)$]

$$= \frac{3(x+3)-2(x+5)+(5x+19)}{(x+5)(x+3)}$$ [each fraction expressed in terms of this L.C.M.]

$$= \frac{3x+9-2x-10+5x+19}{(x+5)(x+3)}$$ [remove brackets on top]

$$= \frac{6x+18}{(x+5)(x+3)}$$ [factorise the top]

$$= \frac{6(x+3)}{(x+5)(x+3)}$$ [factorise the top]

$$= \frac{6}{x+5}$$ [divide top and bottom by $(x+3)$]

Express each of the following as one fraction in its lowest terms:

27. $\dfrac{5x-6}{x^2+x-6}-\dfrac{3}{x+3}$ **28.** $\dfrac{2x}{x^2-1}-\dfrac{1}{x-1}$ **29.** $\dfrac{4}{4-a^2}-\dfrac{1}{2-a}$

30. $\dfrac{5}{2x-3}-\dfrac{3}{2x^2-3x}-\dfrac{1}{x}$ **31.** $\dfrac{a}{ab+b^2}-\dfrac{b}{a^2+ab}$ **32.** $\dfrac{2}{x}-\dfrac{4}{x^2+2x}-\dfrac{1}{x+2}$

Show that each of the following reduces to a constant and find that constant:

33. $\dfrac{x-2}{x-3}+\dfrac{1}{3-x}$ **34.** $\dfrac{2x-3}{x-2}+\dfrac{1}{2-x}$ **35.** $\dfrac{5x-3}{3x-2}-\dfrac{x-1}{2-3x}$

36. $\dfrac{1}{1-x}+\dfrac{x}{x-1}$ **37.** $\dfrac{4x-7}{x-2}+\dfrac{1}{2-x}$ **38.** $\dfrac{x-2}{x^2+2x}+\dfrac{3}{x^2+3x}-\dfrac{x+4}{x^2+5x+6}$

Simplify each of the following:

39. $\dfrac{x+y}{\frac{1}{x}+\frac{1}{y}}$ **40.** $\dfrac{\frac{a}{b}-\frac{b}{a}}{\frac{1}{b}-\frac{1}{a}}$ **41.** $\dfrac{1-\frac{3}{x}}{x-\frac{9}{x}}$ **42.** $\dfrac{x-\frac{2}{x+1}}{\frac{2x}{x+1}-1}$

43. $\dfrac{\frac{x+1}{x-1}-\frac{x-1}{x+1}}{\frac{1}{x+1}+\frac{1}{x-1}}$ **44.** $\left(x+\dfrac{1}{x}\right)^2-\left(x-\dfrac{1}{x}\right)^2$ **45.** $\left(\dfrac{1+a^2}{1-a^2}\right)^2-\left(\dfrac{2a}{1-a^2}\right)^2$

46. Let $f(x)=\dfrac{x^3-1}{x^2-1}$, $x\neq\pm1$, and $g(x)=\dfrac{x^2+x+1}{x^2-x-2}$, $x\neq-1,2$,
if $f(x)\div g(x)=ax+b$, find the value of a and b.

47. If $f(x)=\dfrac{1}{x}$, show that $f(p)-f(q)=f\left(\dfrac{pq}{q-p}\right)$.

48. Simplify $(x+a)^3+(x-a)^3$ and then factorise your simplified expression.

Changing the Subject of a Formula

When we rearrange a formula so that one of the variables is given in terms of the others we are said to be '**changing the subject of the formula**'. The rules in changing the subject of a formula are the same as when solving an equation, that is we can:

1. **Add** or **subtract** the same quantity to both sides.
   (In practice this involves moving a term from one side to another and changing its sign.)
2. **Multiply** or **divide** both sides by the same quantity.
3. **Square** both sides, **cube** both sides, etc.
4. Take the **square root** of both sides, take the **cube root** of both sides, etc.

Note: Whatever letter comes after the word 'express' is to be on its own.

Example

(i) If $\frac{1}{b} + \frac{1}{a} = \frac{1}{c}$, express c in terms of a and b.

(ii) $\sqrt[3]{\frac{3p-2}{2p+1}} = q$, express p in terms of q.

Solution:

(i)
$$\frac{1}{b} + \frac{1}{a} = \frac{1}{c}$$

$ac + bc = ab$ [multiply each term by abc to remove fractions]

$c(a+b) = ab$ [take out common factor c on the left-hand side]

$c = \frac{ab}{a+b}$ [divide both sides by $(a+b)$]

(ii)
$$\sqrt[3]{\frac{3p-2}{2p+1}} = q$$

$\left(\frac{3p-2}{2p+1}\right)^{1/3} = q$ [replace $\sqrt[3]{\ }$ with $(\ )^{1/3}$]

$\left[\left(\frac{3p-2}{2p+1}\right)^{1/3}\right]^3 = (q)^3$ [cube both sides]

$\frac{3p-2}{2p+1} = q^3$ [$(x^{1/3})^3 = x^{1/3\times 3} = x^1 = x$]

$3p - 2 = (2p+1)q^3$ [multiply both sides by $(2p+1)$]

$3p - 2 = 2pq^3 + q^3$ [remove brackets]

$3p - 2pq^3 = q^3 + 2$ [terms with p on the left-hand side]

$p(3 - 2q^3) = q^3 + 2$ [take out common factor p on the left-hand side]

$p = \frac{q^3+2}{3-2q^3}$ [divide both sides by $(3 - 2q^3)$]

Exercise 1.2

1. If $\frac{2a-b}{3} = c$, express a in terms of b and c.
2. If $p - \frac{t}{q} = r$, express q in terms of p, t and r.
3. If $\frac{a}{b} = \frac{b}{c} + d$, express c in terms of a, b and d.
4. If $r = \frac{q^2 - pr}{q+p}$, express p in terms of q and r.

5. If $x = \dfrac{2t-1}{t-1}$, express t in terms of x.

6. If $\sqrt{5x-2} = y$, express x in terms of y.

7. If $\sqrt{\dfrac{y+1}{y-1}} = x$, express y in terms of x.

8. If $\dfrac{1}{u} + \dfrac{1}{v} = \dfrac{1}{f}$, express f in terms of u and v.

9. If $t = k\sqrt{\dfrac{l}{g}}$, express l in terms of t, k and g.

10. If $\dfrac{p}{2} = \sqrt{\dfrac{1}{x^2-1}}$, express x^2 in terms of p.

11. If $px - b = a - qx$, express x in terms of a, b, p and q.
    If $\sqrt{2p} = 4a$ and $q = -8b^2$, show that $8x = \dfrac{1}{a-b}$.

12. If $\sqrt[3]{x+1} = y$, express x in terms of y.

13. If $\sqrt[3]{\dfrac{ax^2}{1-r}} = y$, express x in terms of y, a and r.

14. If $p = \sqrt{\dfrac{q^2-2r^2}{2q^2+r^2}}$, express r in terms of p and q.

15. If $p = q + \sqrt{q^2-1}$, express q in terms of p.

Undetermined Coefficients

When two expressions in x (or any other variable) are equal to one another for all values of x, we ca
equate the coefficients of the same powers of x in the two expressions. This is known as the '**princip
of undetermined coefficients**'.

Method:

1. Remove all fractions and brackets.
2. Form equations by equating coefficients of like terms.
3. Solve the equations to find the coefficients.

Example

(i) If $a(x+b)^2+c=2x^2+12x+23$, for all x, find the value of a, of b and of c.

(ii) If $(ax+k)(x^2-px+1)=ax^3+bx+c$, for all x, show that $c^2=a(a-b)$.

Solution:

(i) Expand the left-hand side and equate coefficients.

$$a(x+b)^2+c=2x^2+12x+23$$
$$a(x^2+2bx+b^2)+c=2x^2+12x+23$$
$$ax^2+2abx+ab^2+c=2x^2+12x+23$$
$$(a)x^2+(2ab)x+(ab^2+c)=2x^2+12x+23$$

Equating coefficients of like terms:

$a=2$ ①

put $a=2$ into ②

$2ab=12$ ②

$2(2)b=12$

$4b=12$

$b=3$

put $a=2$ and $b=3$ into ③

$ab^2+c=23$ ③

$2(3)^2+c=23$

$18+c=23$

$c=5$

$\therefore\ a=2,\ b=3$ and $c=5$

(ii) Expand the left-hand side and equate coefficients.

$$(ax+k)(x^2-px+1)=ax^3+bx+c$$
$$ax^3-apx^2+ax+kx^2-kpx+k=ax^3+ox^2+bx+c \qquad \text{(put in } ox^2\text{)}$$
$$ax^3+(-ap+k)x^2+(a-kp)x+k=ax^3+ox^2+bx+c$$

Equating coefficients of like terms:
(Basic idea is to remove the constants **not** in the answer required)

$-ap+k=0$ ① $\qquad a-kp=b$ ② $\qquad k=c$ ③

From ③, $k=c$. Replace k with c in ① and ②, as k is not in the answer required.

$-ap+k=0$ ①

$-ap+c=0$ ④

$a-kp=b$ ②

$a-cp=b$ ⑤

What we do next is get p on its own from ④ and put this in ⑤.
This removes p which is not in the answer.

$-ap+c=0$ ④

$-ap=-c$

$ap=c$

$p=\dfrac{c}{a}$

$a-cp=b$ ⑤

$a-c\left(\dfrac{c}{a}\right)=b$

$a-\dfrac{c^2}{a}=b$

$a^2-c^2=ab$

$-c^2=ab-a^2$

$c^2=a^2-ab$

$c^2=a(a-b)$

Exercise 1.3

1. If $3ax + 5aby = 12x + 40y$ for all values of x and y, find the value of a and b.

2. $a(2x + 3) + b(x - 4) = 4x + 17$ for all values of x. Write two equations in a and b.
   Hence, or otherwise, find the value of a and the value of b.

3. $3(x^2 + 2x) + 7 = p(x^2 + 2) + qx(x - 3) + r$ for all values of x.
   Find the value of p, the value of q and the value of r.

4. $2x(x + 3) = a(x^2 + 1) + b(x^2 - x) + c$ for all values of x.
   Find the value of a, the value of b and the value of c.

5. $(x + 2)(x^2 + px + q) = x^3 + 5x^2 + 2x - 8$ for all values of x. Find the value of p and the value of q.

6. $(x + a)(2x^2 + bx + 1) = 2x^3 + x^2 - 14x + 3$ for all values of x. Find the value of a and the value of b.

7. $p(x + 1)(x + 2) + q(x + 1) + r = 3x^2 + 5x + 7$ for all values of x.
   Find the value of p, the value of q and the value of r.

8. If $2x^2 + 12x + 13 = p(x + q)^2 + r$ for all x, find the value of p, of q and of r.

9. If $n^2 - 4 = a(n - 1)(n - 2) + b(n - 1) + c$, for all values of n, find the value of a, the value of b and the value of c.

10. $(2x + k)(px + q) = x(2px + 2q - p) + r$, for all x. Show that: **(i)** $k = -1$ **(ii)** $q + r = 0$.

11. $(4x + r)(x^2 + s) = 4x^3 + px^2 + qx + 2$, for all x. Show that $pq = 8$.

12. $(ax + k)(x^2 - px + 1) = ax^3 + bx + c$, for all x. Show that:
    (i) $k = c$ **(ii)** $c = ap$ **(iii)** $b = a(1 - p^2)$.

13. Show that $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.
    If $x^3 + px^2 + qx + r = (x + h)^3$ for all x, show that: **(i)** $p^2 = 3q$ **(ii)** $q^3 = 27r^2$.

Surds

Properties of Surds:

1. $\sqrt{ab} = \sqrt{a}\sqrt{b}$ **2.** $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ **3.** $\sqrt{a}\sqrt{a} = a$

Simplification of Surds

The key idea is to find the largest possible perfect square number greater than 1 that will divide evenly into the number under the square root and use property 1.

The perfect squares greater than 1 are 4, 9, 16, 25, 36, 49, 64, 81, 100, …, etc.

Example

Write each of the following in the form of $a\sqrt{b}$, where b is prime:

(i) $\sqrt{32}$ (ii) $\sqrt{45}$ (iii) $\sqrt{75}$

Express in the form $\frac{a}{b}$, $a, b \in \mathbf{N}$: (iv) $\sqrt{\frac{9}{4}}$ (v) $\sqrt{2\frac{7}{9}}$

(vi) Express $\frac{10}{\sqrt{2}}$ in the form $k\sqrt{2}$.

Solution:

(i) $\sqrt{32} = \sqrt{16 \times 2} = \sqrt{16}\sqrt{2} = 4\sqrt{2}$

(ii) $\sqrt{45} = \sqrt{9 \times 5} = \sqrt{9}\sqrt{5} = 3\sqrt{5}$

(iii) $\sqrt{75} = \sqrt{25 \times 3} = \sqrt{25}\sqrt{3} = 5\sqrt{3}$

(iv) $\sqrt{\frac{9}{4}} = \frac{\sqrt{9}}{\sqrt{4}} = \frac{3}{2}$

(v) $\sqrt{2\frac{7}{9}} = \sqrt{\frac{25}{9}} = \frac{\sqrt{25}}{\sqrt{9}} = \frac{5}{3}$

(vi) $\frac{10}{\sqrt{2}} = \frac{10}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$ (multiply top and bottom by $\sqrt{2}$)

$= \frac{10\sqrt{2}}{2} = 5\sqrt{2}$

Note: The process of removing a surd from the denominator of an expression is called **'rationalising the denominator'**.

Addition and Subtraction

Only like surds can be added or subtracted. Express each surd in its simplest form and add or subtract like surds.

Example

(i) Express $\sqrt{18} + \sqrt{50} - \sqrt{8}$ in the form $a\sqrt{b}$, where b is prime.

(ii) $\sqrt{20} - \sqrt{5} + \sqrt{45} = k\sqrt{5}$; find the value of k.

Solution:

(i) $\sqrt{18} + \sqrt{50} - \sqrt{8}$

$= 3\sqrt{2} + 5\sqrt{2} - 2\sqrt{2}$

$= 8\sqrt{2} - 2\sqrt{2}$

$= 6\sqrt{2}$

(ii) $\sqrt{20} - \sqrt{5} + \sqrt{45}$

$= 2\sqrt{5} - \sqrt{5} + 3\sqrt{5}$

$= 5\sqrt{5} - \sqrt{5}$

$= 4\sqrt{5} = k\sqrt{5}$

$\therefore\ k = 4$

Conjugate Surds

$a + \sqrt{b}$ is a compound surd. Its conjugate is $a - \sqrt{b}$ or $-a + \sqrt{b}$.
$\sqrt{a} - \sqrt{b}$ is a compound surd. Its conjugate is $\sqrt{a} + \sqrt{b}$ or $-\sqrt{a} - \sqrt{b}$.
They have the same components, with one of the signs changed.
The product of a surd and its conjugate is always a rational number.
Think of the difference of two squares: $(a - b)(a + b) = a^2 - b^2$, which is rational.

Example

Show that $\dfrac{-1 + \sqrt{3}}{1 + \sqrt{3}} = 2 - \sqrt{3}$

Solution:

$$\frac{-1 + \sqrt{3}}{1 + \sqrt{3}} = \frac{-1 + \sqrt{3}}{1 + \sqrt{3}} \times \frac{1 - \sqrt{3}}{1 - \sqrt{3}}$$ (multiply top and bottom by $1 - \sqrt{3}$, the conjugate surd of $1 + \sqrt{3}$)

$$= \frac{-1 + \sqrt{3} + \sqrt{3} - 3}{1 - \sqrt{3} + \sqrt{3} - 3}$$ (multiply top by the top and bottom by the bottom)

$$= \frac{-4 + 2\sqrt{3}}{-2}$$ (simplify top and bottom)

$$= 2 - \sqrt{3}$$ (divide each part on top by -2)

Note: We could have also multiplied the top and bottom by $-1 + \sqrt{3}$, also the conjugate surd of $1 + \sqrt{3}$

Exercise 1.4

Express each of the following in the form $a\sqrt{b}$, where b is prime:

1. $\sqrt{12}$ **2.** $\sqrt{18}$ **3.** $\sqrt{20}$ **4.** $\sqrt{72}$ **5.** $\sqrt{48}$ **6.** $\sqrt{4}$

7. $\sqrt{125}$ **8.** $\sqrt{63}$ **9.** $\sqrt{500}$ **10.** $\frac{1}{2}\sqrt{80}$ **11.** $\frac{1}{3}\sqrt{108}$ **12.** $\frac{3}{5}\sqrt{7}$

Express each of the following in the form $\dfrac{p}{q}$, $p, q \in \mathbf{N}$:

13. $\sqrt{\dfrac{4}{9}}$ **14.** $\sqrt{\dfrac{36}{49}}$ **15.** $\sqrt{\dfrac{100}{81}}$ **16.** $\sqrt{2\frac{1}{4}}$ **17.** $\sqrt{1\frac{9}{16}}$ **18.** $\sqrt{4\frac{21}{2}}$

Express each of the following in the form $a\sqrt{b}$, where b is prime:

19. $\dfrac{12}{\sqrt{3}}$ **20.** $\dfrac{6}{\sqrt{2}}$ **21.** $\dfrac{15}{\sqrt{5}}$ **22.** $\dfrac{28}{\sqrt{7}}$ **23.** $\dfrac{12}{\sqrt{18}}$ **24.** $\dfrac{60}{\sqrt{8}}$

Express each of the following in the form $\dfrac{a\sqrt{b}}{c}$, where b is prime:

25. $\dfrac{5}{\sqrt{2}}$ **26.** $\dfrac{4}{\sqrt{3}}$ **27.** $\dfrac{6}{\sqrt{8}}$ **28.** $\dfrac{15}{2\sqrt{5}}$ **29.** $\dfrac{8}{\sqrt{18}}$ **30.** $\dfrac{25}{\sqrt{4}}$

31. $\sqrt{50} - \sqrt{200} + \sqrt{98} = k\sqrt{2}$. Find the value of k^2.

32. Express $(2 - \sqrt{3})^2$ in the form $a + b\sqrt{c}$, where $a, b, c \in \mathbf{Z}$.

33. Express $(7 + \sqrt{5})^2 - (7 - \sqrt{5})^2$ in the form $k\sqrt{5}$, $\quad k \in \mathbf{N}$.

34. Express $\dfrac{1}{3\sqrt{5}} - \dfrac{1}{2\sqrt{20}}$ in the form $k\sqrt{5}$, $\quad k \in \mathbf{Q}$.

35. Show that:

(i) $\dfrac{1}{\sqrt{2}+1} = \sqrt{2} - 1$

(ii) $\dfrac{1}{2-\sqrt{3}} = 2 + \sqrt{3}$

(iii) $\dfrac{\sqrt{2}+1}{\sqrt{2}-1} = 3 + 2\sqrt{2}$

(iv) $\dfrac{\sqrt{5}-1}{\sqrt{5}-2} = 3 + \sqrt{5}$

(v) $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}} = 5 + 2\sqrt{6}$

(vi) $\dfrac{\sqrt{5}-1}{3-\sqrt{5}} = \dfrac{1+\sqrt{5}}{2}$

(vii) $\dfrac{1}{\sqrt{3}-1} + \dfrac{1}{\sqrt{3}+1} = \sqrt{3}$

(viii) $\dfrac{1}{\sqrt{2}-1} - \dfrac{1}{\sqrt{2}+1} = 2$

36. Express $\dfrac{1}{a} - \dfrac{1}{b}$ as a single fraction.

Hence, or otherwise, express $\dfrac{1}{a} - \dfrac{1}{b}$ in the form $-k\sqrt{k}$, when $a = 1 - \sqrt{2}$ and $b = 1 + \sqrt{2}$.

37. Factorise $x^2 - y^2$.

Hence, or otherwise, evaluate $\sqrt{x^2 - y^2}$, when $x = \left(\sqrt{a} + \dfrac{1}{\sqrt{a}}\right)$ and $y = \left(\sqrt{a} - \dfrac{1}{\sqrt{a}}\right)$, $a > 0$.

Simultaneous Linear Equations

Simultaneous linear equations in two variables are solved with the following steps:

1. Write both equations in the form $ax + by = k$ and label the equations ① and ②.
2. Multiply one or both of the equations by a number in order to make the coefficients of x or y the same, but of opposite sign.
3. Add to remove the variable with equal coefficients but of opposite sign.
4. Solve the resultant equation to find the value of the remaining unknown (x or y).
5. Substitute this value in equation ① or ② to find the value of the other unknown.

Solution Containing Fractions

If the solution contains fractions the substitution can be difficult.
In such cases the following method is useful:

1. Eliminate y and find x. **2.** Eliminate x and find y.

Example

Solve for x and y the simultaneous equations

$$\frac{x+1}{2}-\frac{y+3}{3}=4, \quad x+\frac{y-3}{2}=\frac{1}{2}$$

Solution:

Write each equation in the form $ax + by = k$ and label the equations ① and ②.

$$\frac{x+1}{2}-\frac{y+3}{3}=4$$

$$\frac{6(x+1)}{2}-\frac{6(y+3)}{3}=6(4)$$

(multiply each part by 6)

$$3(x+1)-2(y+3)=24$$
$$3x+3-2y-6=24$$
$$3x-2y=27 \quad ①$$

$$x+\frac{y-3}{2}=\frac{1}{2}$$

$$2(x)+\frac{2(y-3)}{2}=2(\tfrac{1}{2})$$

(multiply each part by 2)

$$2x+(y-3)=1$$
$$2x+y-3=1$$
$$2x+y=4 \quad ②$$

Now solve between equations ① and ②:

$$3x-2y=27 \quad ①$$
$$4x+2y=8 \quad ②\times 2$$
$$7x=35 \quad \text{(add)}$$
$$x=5$$

(put $x = 5$ into ① or ②)

$$2x+y=4 \quad ②$$
$$2(5)+y=4$$
$$10+y=4$$
$$y=-6$$

Thus, $x = 5$ and $y = -6$.

Simultaneous linear equations in three variables are solved with the following steps:

1. Write all three equations in the form $ax + by + cz = k$ and label the equations ①, ② and ③.
2. Select one pair of equations and eliminate one of the variables; call this equation ④.
3. Select another pair of equations and eliminate the **same** variable; call this equation ⑤.
4. Solve the equations ④ and ⑤.
5. Put the answers from step 4 into ① or ② or ③ to find the value of the third variable.

Example

Solve for x, y and z:

$$x+2y+z=3$$
$$5x-3y+2z=19$$
$$3x+2y-3z=-5$$

Solution:

All three equations are in the form $ax + by + cz = k$.

Label the equations ①, ② and ③.

Eliminate z from two different pairs of equations.

$$x + 2y + z = 3 \quad ①$$
$$5x - 3y + 2z = 19 \quad ②$$
$$3x + 2y - 3z = -5 \quad ③$$

$$\begin{array}{rl} -2x - 4y - 2z = -6 & ① \times -2 \\ 5x - 3y + 2z = 19 & ② \\ \hline 3x - 7y = 13 & ④ \text{ (add)} \end{array}$$

$$\begin{array}{rl} 3x + 6y + 3z = 9 & ① \times 3 \\ 3x + 2y - 3z = -5 & ③ \\ \hline 6x + 8y = 4 & ⑤ \end{array}$$

Now solve between equations ④ and ⑤ to find the values of x and y.

$$\begin{array}{rl} -6x + 14y = -26 & ④ \times -2 \\ 6x + 8y = 4 & ⑤ \\ \hline 22y = -22 & \\ y = -1 & \end{array}$$

(put $y = -1$ into ④ or ⑤)

$$\begin{array}{rl} 3x - 7y = 13 & ④ \\ 3x - 7(-1) = 13 & \\ 3x + 7 = 13 & \\ 3x = 6 & \\ x = 2 & \end{array}$$

Now put $x = 2$ and $y = -1$ into ① or ② or ③ to find the value of z.

$$x + 2y + z = 3 \quad ①$$
$$(2) + 2(-1) + z = 3 \qquad \text{(put in } x = 2 \text{ and } y = -1)$$
$$2 - 2 + z = 3$$
$$z = 3$$

Thus, $x = 2$, $y = -1$ and $z = 3$.

Note: Any of the variables x, y or z could have been eliminated at the beginning.

If one equation contains only two unknowns, then the other two equations should be used to obtain a second equation in the same two unknowns, e.g. solve:

$3x + 2y - z = 3$ ① $\qquad$ $5x - 3y + 2z = 3$ ② $\qquad$ $5x + 3z = 14$ ③

Here, from equations ① and ②, y should be eliminated to obtain an equation in x and z, which should then be taken with equation ③.

Exercise 1.5

Solve for x and y:

1. $3x + 2y = 9$
   $x - y = -2$

2. $4x + 3y = -23$
   $x + 2y = -12$

3. $5x + 4y = 22$
   $3x + 5y = 21$

4. $x = 5 - y$
   $\frac{4x}{3} + 8 = \frac{y}{2}$

5. $3x - 2y = 19$
   $\frac{x}{3} + \frac{y}{2} = 5$

6. $2x + y = 3(y - x) + 7$
   $\frac{x}{3} = 2 - \frac{y}{4}$

7. $\dfrac{2x-5}{3}+\dfrac{y}{5}=6$
$\dfrac{3x}{10}+2=\dfrac{3y-5}{2}$

8. $\dfrac{3x}{5}-\dfrac{y}{4}=8$
$\dfrac{2x}{3}=13-\dfrac{3y}{4}$

9. $2x+3y=-2$
$3x+7y=-6$

Solve for x, y and z:

10. $3x+5y-z=-3$
$2x+y-3z=-9$
$x+3y+2z=7$

11. $2x+3y-z=-7$
$5x-2y-4z=3$
$3x+y+2z=-7$

12. $3x-y+3z=1$
$x+2y-2z=-1$
$4x-y+5z=4$

13. $x+y-z=0$
$x-y+z=4$
$x-y-z=-8$

14. $x+2y-z=-1$
$2x+y+3z=14$
$3x-y-z=-14$

15. $2x+y-z=-3$
$x+3y+2z=1$
$3x-2y+z=10$

16. $2x+y-z=-3$
$x+3y+2z=1$
$3x-y=9$

17. $x+y+z=1$
$2x-3y-2z=-9$
$2x-3z=-16$

18. $x+y=-1$
$y+3z=-11$
$3x+5z=-12$

19. **(i)** Solve the simultaneous equations:
$3x+y+z=16$
$2x-y+3z=24$
$x-y-z=0$

(ii) Hence, or otherwise, solve:
$3a^2+(b-2)+(2c-1)=16$
$2a^2-(b-2)+3(2c-1)=24$
$a^2-(b-2)-(2c-1)=0$

20. If the curve $y=ax^2+bx+c$ contains the points $(0, 5)$, $(1, 4)$ and $(-1, 10)$ find the value of a, th value of b and the value of c.

21. $f(x)=px^2+qx+r$. If $f(-2)=7$, $f(1)=-2$ and $f(2)=3$, find the value of p, the value of q and th value of r.